Methodus generalis, aequationes differentiarum partialium, ...

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Pfaff method. nova integrandi

Methodus generalis,

aequationes differentiarum partialium,

nec non

aequationes differentiales vulgares,

utrasque primi ordinis, inter quotcunque variabiles, complete integrandi.

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§. 1.

Inter insignia inventa, quibus sagacissimus La Grange Analysin auxit, referenda est methodus generalis, aequationes differentiarum partialium inter tres variabiles integrandi. Cujus integrationis investigationem difficultate haud caruisse, vel exinde intelligere licet, quod Eulerus, peritissimus calculi artifex, in longa harum aequationum pertractatione (Calc. Integr. Vol. III. p. 37 — 178.) casus tantum particulares, non ratione uniformi, sed variis artificiis usus, solutos dederit, ipse confessus (p. 132.): „se longissime adhuc a solutione problematis generalis distare." Quum quidem Eulerus notionem haud satis generalem genuinae harum aequationum indolis mente concepisse videatur, La Grangius methodum suam exinde deduxit, quod easdem ex principio uniformi et generali contemplatus fuerit. Sit nimirum z functio variabilium x et y, $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$, tum vulgo constat, aequationem differentiarum partialium inter tres variabiles nil aliud esse, quam relationem datam inter p, q, x, y, z, ex qua quaeritur relatio inter x, y, z. Jam cum sit $dz = pdx + qdy$, La Grange considerat hanc aequationem ceu aequationem differentialem inter tres quantitates x, y, z, quae praeterea quantitatem p tanquam functionem indeterminatam ipsarum trium variabilium involvit. At vero constat, ut talis aequatio integrabilis sit, seu certam inter tres variabiles aequationem finitam inferat, illius coefficientes non ad lubitum assumi posse, sed certam inter eos relationem requiri. Ex hoc criterio integrabilitatis sive realitatis (Euler l. c. p. 5. 6.) petenda est determinatio coefficientis p, tanquam functionis τῶν x, y, z. Quo coefficiente rite determinato ipsa deinceps integratio aequationis $dz = pdx + qdy$, secundum regulas aliunde cognitas, praebet relationem quaesitam inter x, y, z. Quod nunc determinationem τȣ̃ p attinet, praedicta conditio integrabilitatis denuo deducit ad aequationem differentiarum partialium inter quatuor variabiles, quae autem est linearis, quotientes differentiales in prima tantum dimensione continens, quamque igitur ex principiis aliunde a La Grangio demonstratis (Mémoires de l'Acad. de Berlin 1774. 1779.) integrare licet. In hac solutione restabat difficultas, a qua diu se vexatum fuisse, ipse La Grangius confitetur [a]). Cum nimirum p ceu functio trium variabilium x, y, z, detur per aequationem differentiarum partialium, hujus integratio ex lege nota involvit functionem arbitrariam duarum quantitatum: quum tamen determinatio τȣ̃ z, tanquam functionis duarum variabilium x et y, functionem arbitrariam unius tantum quantitatis admittat. Hanc ipsam difficultatem atque contraditionem apparentem feliciter tandem enodavit La Grange, ostenditque ratiociniis ingeniosis, quomodo functio ista duarum quantitatum, quae natura sua infinities latior est quam functio arbitraria unius quantitatis, ad hujusmodi functionem unius variabilis revocetur.

Ceterum alii Analystae, qui methodum La Grangianam applicationibus illustrarunt, veluti celeberr. Le Gendre (Mémoires de l'Acad. de Paris Année 1787. p. 337.), praedictam difficultatem haud animadverterunt [b]), vel silentio praetermiserunt, forte, quoniam eadem defectus minus essentialis methodi ipsius videri poterat, quippe in applicatione methodi non necessè erat, aequationis auxiliaris pro p integrale completum invenire, (quod functionem arbitrariam duarum quantitatum involvisset), verum valor particularis τȣ̃ p vel ad complete integrandam aequationem ipsam propositam sufficere jam poterat.

Quum praedicta ratione integratio aequationum differentiarum par-

a) Leçons sur le Calcul des fonctions. Nouv. édit. 8. Paris 1806. p. 390. „Cette difficulté, je l'avoue, m'a long-temps tourmenté." cf. p. 386.

b) Quod ipsum observat La Grange (Leçons p. 386.)

tialium primi ordinis inter tres variabiles pro confecta et omnibus numeris absoluta sit habenda, aliter res se habet, si quaestio agitur de quatuor vel pluribus variabilibus. Eulerus, qui aequationes inter tres variabiles, sive, quod perinde est, investigationem functionem duarum variabilium, ample pertractaverat, pro quatuor variabilibus pauca tantum exempla resolvit (l. c. p. 423—41), in quibus ipse non nisi prima elementa hujus scientiae contineri observat, et casum quinque variabilium ob penuriam materiae quae sunt viri summi verba p. 457.), ne attingere quidem voluit. Ipse deinceps La Grange (Mém. de Berl. 1774. 79.), aliique Analystae, veluti Monge (Mém. de Paris 1784. p. 556.), et le Gendre (l. c.), casus duntaxat valde limitatos in hoc genere contemplati sunt, qui quidem facile vel ad aequationes inter tres variabiles vel ad aequationes lineares reduci possunt, quam formam simplicissimam pro quotcunque variabilibus integrare docuerat La Grange (Mém. l. c.).

Quod si quidem methodum modo antea laudatum La Grangianam, aequationes differentiarum partialium inter tres variabiles generatim et complete integrandi, ad plures variabiles extendere conemur, mox ad inextricabiles difficultates delabimur: unde forte accidit, ut Analystae hactenus (quantum equidem sciam) hanc applicationem nondum tentaverint. Quibus permotus difficultatibus equidem satius duxi, totam aequationes differentiarum partialium considerandi rationem, ex qua methodus La Grangiana originem duxit, deserere, atque aliud principium in auxilium vocare, ex quo, etiam si per se simplicissimo, hactenus tamen istae aequationes nondum consideratae fuerunt, quodque revera, nisi cum aliis subsidiis jungeretur, parum esset frugiferum. Aequationes nimirum differentiarum partialium contemplari licet tanquam aequationes differentiales vulgaris generis truncatas inter plures variabiles, quam quae principaliter occurrunt, ipsis scilicet quotientibus differentialibus (p, q, etc.) variabilium loco habitis, quarum differentialia (dp, dq, ..) ideo desunt, quoniam ea in zero ducta esse censentur. Ita aequatio $dz = pdx + qdy = pdx + qdy + o.dp$ est aequatio inter quatuor variabiles x, y, z, p; q enim est quantitas a reliquis quatuor quantitatibus, ex hypothesi, data ratione dependens. Sic in genere aequatio differentiarum partialium inter m variabiles considerari potest tanquam aequatio differentialis vulgaris inter $2m-2$ variabiles, si m variabilibus principalibus (z, x, y, ...) adjiciantur $m-2$ variabiles accessoriae (p, q, ...), cum occurrant $m-1$ quotientes differentiales, quorum autem unus, per reliquos quantitates datus, non in computum venit. Jam vero Mongius (l. c) jam pridem docuit, contra opinionem antea vulgo receptam, aequationes differentiales, quae criteriis sic dictis integrabilitatis haud satisfaciant, haud pro absurdis habendas esse, sed potius easdem revera integrationem admittere, modo non per unam aequationem finitam, verum per systema plurium aequationum [c]). Quae egregia observatio si in nostro problemate adhibeatur, considerandum est, quod solutio aequationis differentiarum partialium essentialiter exigat expressionem unius variabilium principalium per reliquas. Quare variabiles accessoriae (p, q, etc.) in computum ingressae iterum sunt eliminandae: quarum numerus cum sit $m-2$, numerus aequationum, quarum systemate integratio continetur, non major esse debet quam $m-1$; quod cum ita fuerit, ex his $m-1$ aequationibus eliminando variabiles accessorias remanet aequatio finalis inter ipsas variabiles principales. At in hoc ipso cardo difficultatis versatur. Namque sicuti evidens est, aequationem differentialem inter tres variabiles per systema duarum aequationum finitarum semper esse integrabilem, ita haud difficulter intelligitur, idemque a Mongio explicatum est (l. c. p. 533, 34.), aequationem differentialem inter n variabiles regulariter, salvis exceptionibus singularibus, per systema $n-1$ aequationum integrari posse. Itaque in nostro casu loco desideratarum $m-1$ aequationum nancisceremur $2m-3$ aequationes integrales, i. e. justo plures, quibus finem propositum neutiquam assequi liceret. Sic igitur nostra aequationes differentiarum partialium considerandi ratio sterilis omnino foret, neque aliud quidquam suppeditare videretur, quam reductionem problematis simplicioris ad problema magis complicatum. Etenim quanquam aequationes differentiarum partialium rite considerentur tanquam aequationes differentiales vulgares, ex altera tamen parte hae posteriores longe latius patent, et contra illae harum formam tantum simplicissimam ostendunt. Quare opinio veri aliqua specie haud carere videretur, quod forma simplicissima ceu casus singularis exceptus per systema pauciorum aequationum integrari

c) Haud congrua sunt, quae Eulerus de aequationibus differentialibus inter tres variabiles criterio integrabilitatis non respondentibus statuit, „quod eae sint absurdae, nihil plane significantes, quodque de earum integratione ne cogitari quidem possit (l. c. p. 7. 8.). Acute observat Monge (Mém. de Paris 1784. p. 535.): Ce qu'il y avoit d'absurde, c'étoit que leurs intégrales pussent être exprimées par une seule équation."

queat, quam forma generalis [d]). Attamen rem secus se habere, accuratior consideratio aequationum differentialium vulgarium inter quotcunque variabiles me docucit, sicque perveni ad propositionem novam et mihi quidem inexpectatam, quod quaevis aequatio differentialis vulgaris primi ordinis inter 2n et 2n — 1 variabiles semper per systema n aequationum (vel pauciorum) integrari possit. Ex qua propositione generali, quae naturam istarum aequationum hactenus etiam post laudatam Mongii observationem haud satis perspectam magis illustrare videtur, sponte quoque consequitur ceu corollarium particulare, solutio aequationum differentiarum partialium inter quotcunque variabiles completa.

Quae hactenus universe adumbrata clariorem lucem accipient, si ea primum ad casum simplicissimum, scilicet aequationes differentiarum partialium inter tres variabiles, applicemus. Et enim quanquam hoc problema jam a La Grangio solutum sit, ipsa tamen expositio inventoris (Leçons l. c.) a dubiis non omnino libera, novaque egregiae hujus solutionis deductio et illustratio haud prorsus superflua esse videtur, quae quidem abstrahendo a criterio sic dicto integrabilitatis, nec aliunde supposita jam integratione aequationum linearium, transformatione tantum simplici aequationis propositae generalis breviter absolvitur: in qua porro deductione phaenomenon singulare, La Grangii sagacitate detectum, ab ipso tantum ad enodandam difficultatem solutioni adhaerentem in auxilium vocatum, quodque accessorium ac veluti accidentale videri poterat, jam directe et principaliter investigatum, atque ex ipsis calculi fontibus haustum pro fundamento totius solutionis aequationum differentiarum partialium inter tres variabiles ponitur.

§. 2.

Priusquam autem rem ipsam aggrediamur, praemonendum est, in omni hac disquisitione supponi tanquam cognitam integrationem aequationum differentialium inter duas variabiles, quam saltem per approximationem ope serierum in potestate esse constat. Quae quidem integratio ab omnibus Analystis, qui aequationes differentiarum partialium tractarunt, postulatur [e]). At vero

d) Dari casus exceptos, ipse Mongius observat l. c. p. 534. „Le nombre des équations intégrales n'est pas toujours, comme dans le cas précédent, égal au nombre des variables diminué d'une unité.

e) Euler Calc. Integr. Vol. III. p. 34. „in hoc negotio, quoties resolutionem ad aequationem differentialem inter duas variabiles reducere licet, problema pro resoluto erit habendum" cf. p. 67. Idem observat La Grange Mém. de Berlin 1779. p. 153.

vero huc non tantum pertinet casus vulgo notus, quo datur una aequatio inter duas variabiles, verum etiam casus complicatior, quo duae, tres etc. in genere n aequationes conjunctim intègrandae inter tres, quatuor . . vel $n+1$ variabiles dantur. Nota sunt, quae Alembertus aliique Analystae de hujusmodi integrationibus docuerunt. Sufficiat, hic breviter ostendere, id, quod alii auctores non satis generaliter illustrarunt, cujusque in sequentibus frequens erit usus, quod tales integrationes semper ad integrationem unius aequationis inter duas variabiles revocari possint.

Sint propositae n aequationes differentiales primi ordinis inter $n+1$ variabiles $z, \overset{1}{x}, \overset{2}{x}, \ldots \overset{n}{x}$ sub hac forma, ad quam semper eas revocare licet:

1) $d\overset{1}{x} = \overset{1}{X}dz$

2) $d\overset{2}{x} = \overset{2}{X}dz$

3) $d\overset{3}{x} = \overset{3}{X}dz$

. . . .

n) $d\overset{n}{x} = \overset{n}{X}dz,$.

ubi quantitates $\overset{1}{X}, \overset{2}{X}, \ldots \overset{n}{X}$ dato utcunque modo pendent ab ipsis variabilibus $z, \overset{1}{x}, \ldots \overset{n}{x}$; tum his aequationibus inferuntur relationes inter istas $n+1$ variabiles, ex quibus una variabili pro principali assumta, reliquas ab illa dependentes tanquam ejusdem functiones considerare licet. Differentiando aequationem primam, dz tanquam differentiale constans assumendo, prodit $d^2\overset{1}{x} = d\overset{1}{X}.dz$. Est autem differentiale completum τῦ $\overset{1}{X}$, tanquam functionis explicitae plurium quantitatum, hujus formae: $d\overset{1}{X} = Md\overset{1}{x} + Nd\overset{2}{x} + \ldots$, quod, substituendo valores τῶν $d\overset{1}{x}, d\overset{2}{x}, d\overset{3}{x}, \ldots$ ex ipsis aequationibus datis, praebet $d\overset{1}{X} = Pdz$, unde fit $d^2\overset{1}{x} = Pdz^2$, ubi P itidem est functio data variabilium $z, \overset{1}{x}, \ldots \overset{n}{x}$. Simili modo hanc aequationem rursus differentiando, provenit $d^3\overset{1}{x} = Qdz^3$; quas differentiationes continuando tandem differentiale n^{tum} prodit $d^n \overset{1}{x} = S dz^n$. Quod si nunc concipiatur, ex his n aequationibus

$$1)\ d\overset{1}{x} = \overset{1}{X} dz$$

$$2)\ d^2\overset{1}{x} = P dz^2$$

$$3)\ d^3\overset{1}{x} = Q dz^3$$

$$\cdots\cdots\cdots$$

$$n)\ d^n\overset{1}{x} = S dz^n$$

$n - 1$ variabiles $\overset{2}{x}, \overset{3}{x}, \ldots \overset{n}{x}$ eliminatas esse, prodibit aequatio differentialis n^{ti} gradus, quae tantum duas variabiles $\overset{1}{x}$ et z involvit. Cujus integratio completa praebet $\overset{1}{x}$ tanquam functionem cognitam variabilis z, in quam ingredientur praeterea n constantes arbitrariae a, b, c, ... Ipsas deinceps reliquas variabiles $\overset{2}{x}, \overset{3}{x}, \ldots \overset{n}{x}$, tanquam functiones itidem cognitas variabilis z, earundemque constantium a, b, c, ..., considerari posse, ex ipsa praedicta eliminatione, valores istarum variabilium suppeditante, manifestum est.

Ceterum dantur casus, quibus integratio plurium aequatiorum differentialium, conjunctim locum habentium, facilius, quam modo generali hic breviter adumbrato, perfici potest.

§. 5.

Problema I.

Aequationem differentiarum partialium inter tres variabiles complete integrare.

Solutio.

Cum in aequatione $dz = p dx + q dy$, relatione data inter q, p, x, y, z, nil inferatur quam determinatio τȣ q per p, x, y, z, quantitas p restat indeterminata, hincque istam aequationem ceu aequationem differentialem inter quatuor variabiles z, x, y, et p considerare licet. Jam loco quantitatum z, x, p, alias tres quantitates a, b, c, introducere licet, dum pro z, x, p pro lubitu assumantur functiones τῶν y, a, b, c. Quomodocunque enim se habeant quantitates z, x et p, (quarum relatio adhuc incognita est), et qualitercunque accipiantur praedictae functiones, semper concipi possunt tres valores τῶν a, b, c, qui assumtis tribus aequationibus valores τῶν z, x, p per y, a, b, c exprimentibus satisfaciant. Qua quidem ratione aequatio proposita $dz = p dx + q dy$ generatim transformabitur in aliam aequationem inter quatuor variabiles y, a, b, c. Jam vero istae functiones substituendae ita sunt definiendae, ut tam y, quam dy ex calculo exeant: tum enim prodibit aequatio inter tres variabiles a, b, c, quam sponte per systema duarum aequationum integrare, sicque (uti in introductione §. 1. observatum est), finem propositum assequi licet.

Cum x, z, et p concipiantur esse functiones τῶν y, a, b, c, earum differentialia sequenti modo exprimi possunt:

$$dx = X dy + \chi da + \chi' db + \chi'' dc,$$
$$dz = Z dy + \zeta da + \zeta' db + \zeta'' dc,$$
$$dp = P dy + \pi da + \pi' db + \pi'' dc;$$

ubi perinde est, sive $X, \chi, \chi', \chi''; Z, \ldots \zeta''; P, \ldots \pi''$, explicite ab y, a, b, c sive implicite ab y, x, z, p pendere censeantur. Cum porro detur relatio inter q, p, x, y, z, concipi potest quantitas q expressa per p, x, y, z, quo facto ejus differentiale hanc nanciscetur formam: $dq = q' dx + q'' dy + q''' dz + q'''' dp$, ubi q', q'', q''', q'''', pro datis functionibus τῶν x, y, z, p haberi possunt.

Quibus praemissis aequatio proposita

$$dz = p dx + q dy$$

hanc induit formam:

$$o = \left.\begin{matrix} Z \\ -pX \\ -q \end{matrix}\right| dy + \left.\begin{matrix} \zeta \\ -p\chi \\ \end{matrix}\right| da + \left.\begin{matrix} \zeta' \\ -p\chi' \\ \end{matrix}\right| db + \left.\begin{matrix} \zeta'' \\ -p\chi'' \\ \end{matrix}\right| dc.$$

Jam ut dy ex haec aequatione exeat, ponendum est

1) $Z = pX + q$

tum superest aequatio:

$$o = da + \frac{\zeta' - p\chi'}{\zeta - p\chi} db + \frac{\zeta'' - p\chi''}{\zeta - p\chi} dc.$$

Deinde ut haec aequatio etiam ab ipsa quantitate y libera fiat, coefficientes $\frac{\zeta' - p\chi'}{\zeta - p\chi}$, $\frac{\zeta'' - p\chi''}{\zeta - p\chi}$, qui generatim ab y, a, b, c, pendent, respectu τῦ y constantes esse debent, hincque tales, ut eorum differentialia secundum y accepta evanescant [f]). Quod si autem $d^y\left(\frac{M}{N}\right)$ ponatur $=0$, erit $Nd^yM - Md^yN = 0$, sive $\frac{d^yM}{M} = \frac{d^yN}{N}$. Hinc esse debet

$$\frac{d^y(\zeta - p\chi)}{\zeta - p\chi} = \frac{d^y(\zeta' - p\chi')}{\zeta' - p\chi'} = \frac{d^y(\zeta'' - p\chi'')}{\zeta'' - p\chi''}.$$

Est autem $d^y(\zeta - p\chi) = d^y\zeta - p\,d^y\chi - \chi\,d^y p$; porro ex theoremate notissimo de differentialibus functionum plurium variabilium, cum sint x, et z functiones τῶν y, a, b, c, erit $d^y\zeta = d^aZ$, $d^y\chi = d^aX$; hinc

$$\begin{aligned} d^y\zeta - p\,d^y\chi = d^aZ - p\,d^aX &= d^a(Z - pX) + X d^a p \\ &= d^a q + X d^a p, \text{ ob } Z - pX = q. \end{aligned}$$

Inde prodit $d^y(\zeta - p\chi) = d^a q + X d^a p - \chi d^y p$.

Jam vero est $d^a q = q' d^a x + q'' d^a y + q''' d^a z + q'''' d^a p$,
vel ob $d^a x = \chi$, $d^a y = 0$, $d^a z = \zeta$, $d^a p = \pi$, erit $d^a q = q'\chi + q'''\zeta + q''''\pi$;
cum praeterea sit $d^y p = P$, prodit $d^y(\zeta - p\chi) =$

$$\left.\begin{matrix} q' \\ -P \end{matrix}\right| \chi + q'''\zeta + \left.\begin{matrix} q'''' \\ +X \end{matrix}\right| \pi, \text{ vel } \frac{d^y(\zeta - p\chi)}{\zeta - p\chi} = \frac{q'''\zeta + \left.\begin{matrix} q' \\ -P \end{matrix}\right| \chi + \left.\begin{matrix} q'''' \\ +X \end{matrix}\right| \pi}{\zeta - p\chi}.$$

Simili modo prodeunt quotientes $\frac{d^y(\zeta' - p\chi')}{\zeta' - p\chi'}$, $\frac{d^y(\zeta'' - p\chi'')}{\zeta'' - p\chi''}$, permutando tantum in expressione inventa ζ, X cum ζ', χ'; ζ'', χ''. Jam evidens est, conditionem praedictam adimpleri, sive tres quotientes aequales fieri, si ponatur

2) $q'''' + X = 0$
3) $q' - P = -p q'''$;

tum enim tres isti quotientes abeunt in q'''.

Hinc prodit $X = -q''''$
$P = q' + p q'''$

Unde invenitur ex aequatione (1)

$$Z = pX + q = q - p q''''.$$

f) Si quantitas ω a pluribus aliis pendens, secundum unam earum e, g, y differentietur, ceteris constantium instar habitis, tum $\frac{d\omega}{dy}$ brevitatis caussa designo per $d^y\omega$.

Quibus igitur valoribus pro X, Z, P suppositis, aequatio transformata tres tantum variabiles a, b, c cum earundem differentialibus continebit. Ita quidem X, Z, P per x, z, p, y dantur, quoniam q, q', q'', q'''', datae sunt functiones harum quantitatum. Quare si in formulis supra assumtis pro dx, dz, dp, quantitates a, b, c constantium instar tractentur, ex tribus aequationibus differentialibus:

$$dx = X dy = -q'''' dy$$
$$dz = Z dy = (q - pq'''') dy$$
$$dp = P dy = (q' + pq'') dy$$

valores τῶν x, z, p, per y integrando exprimi poterunt (§. 2.). Quae expressiones cum ex supra (§. 2.) demonstratis tres quantitates constantes arbitrarias α, β, γ, involvant, quantitates x, z, p etiam ceu functiones τῶν α, β, γ praeter y considerare licet; tumque differentialia completa dx, dz, dp, dum quoque α, β, γ, variabilium instar tractantur, sponte ipsas formas assumtas induent: ubi nunc pro α, β, γ poni posse a, b, c evidens est. Sic igitur tres illas aequationes differentiales integrando revera pro x, z, p, tales functiones τῶν y, a, b, c inventae sunt, quibus substitutis aequatio proposita $dz = pdx + qdy$ transformatur in aequationem tres tantum variabiles a, b, c involventem hujus formae: $da + Bdb = Cdc$, ubi jam B, C pro functionibus cognitis τῶν a, b, c, habenda sunt.

Restat nunc, ut ostendatur, quomodo haec aequatio per systema duarum aequationum integrari possit. Considerata c tanquam constante, integretur aequatio $da + Bdb = 0$, sive, (quod concessa hac integratione semper in potestate esse facile demonstratur), inveniatur multiplicator M, qui formulam $da + Bdb$ aequalem faciat differentiali functionis duarum variabilium a, b. Sit porro haec functio = N, tum erit

1) $N = \varphi c$

denotante φc functionem arbitrariam quantitatis c, quae functio, si c est constans, ipsa etiam constantis locum sustinet. At cum tam multiplicator M, quam integrale N praeter a, b etiam contineat quantitatem c, constantis instar habitam, erit completum differentiale τῦ N, c etiam tanquam variabilem tractando, $dN = Mda + MBdb + \frac{dN}{dc} \cdot dc$, ubi cum N sit functio cognita τῶν a, b, c, etiam $\frac{dN}{dc} = N'$ talis erit functio. Inde complete differentiando aequationem $N = \varphi c$, prodit

$$M da + MB db + N' dc = dc . \varphi' c.$$

Est autem $M da + MB db = MC dc$, hinc fit

$$2)\ MC + N' = \varphi' c.$$

Haec igitur aequatio combinata cum priore (1) praebet integrale completum aequationis inter tres variabiles $da + B db = C dc$.

Quae jam facile ad integrationem aequationis nostrae differentiarum partialium transferri possunt. Quum nimirum z, x, p dentur per y, a, b, c, ope trium aequationum differentialium auxiliarium, vice versa a, b, c considerari possunt tanquam functiones datae τῶν z, x, p, y; sicque etiam M et N' tales erunt functiones. Hinc aequationes (1) et (2) duas relationes inferunt inter quatuor quantitates z, x, p, y; unde, si concipiatur eliminata quantitas p, prodit relatio quaesita inter z, x, y. Haec integratio pro completa est habenda, cum ea complectatur functionem arbitrariam signo φ denotatam.

Denotemus hic et in sequentibus signis $F, \overset{1}{F}, \overset{2}{F}, \overset{3}{F}, \ldots$; porro $f, \overset{1}{f}, \overset{2}{f}, \overset{3}{f}, \ldots$ functiones cognitas unius vel plurium variabilium; signo ψ autem (vel φ) functionem arbitrariam; porro exprimamus $\frac{dfx}{dx}$ per $f'x$, et pro functionibus plurium variabilium, $\frac{df(x, y, z, ..)}{dx}$ vel $d^x f(x, y, z..)$ per $f'x$; $d^y f(x, y, z\ldots)$ per $f'y$; $d^z f(x, y, z\ldots)$ per $f'z$, etc. quam quidem commodam notationem La Grangius in Lectionibus adhibuit (p. 53.).

Quibus praemissis integratio inventa semper sub hac forma exhiberi poterit:

$$1)\ F(z, y, x, p) = \psi[f(z, y, x, p)]$$

$$2)\ \overset{1}{F}(z, y, x, p) = \psi'[f(z, y, x, p)];$$

ubi vix opus est ut moneam, sub signo ψ, functionem signo f denotatam unius quantitatis vicem sustinere, et hoc sensu signum ψ' intelligendum esse.

§. 4.

Aequatio generalis inter quatuor variabiles z, x, y, p haec est:

$$dz = P dx + Q dy + R dp,$$

ubi P, Q, R quascumque functiones datas ipsarum variabilium z, x, y, p denotant. Hujus formae, aequatio priori §. considerata casum tantum particularem sistit, duplici respectu limitatum, primo quod sit $R = 0$, deinde quod functio P, generatim per quatuor variabiles utcunque data, uni variabili p aequalis sumatur. Nihilominus tamen forma etiam ista generalis ad tres variabiles revocari, sicque per systema simile duarum aequationum integrari potest. Quod quidem sequenti problemate ostendetur.

Problema II.

Aequationem differentialem primi ordinis quamcunque inter quatuor variabiles z, x, y, p, $\quad dz = Pdx + Qdy + Rdp$,
denotantibus P, Q, R quascunque functiones datas istarum variabilium, in aequationem inter tres variabiles transformare, eandemque per systema duarum aequationum integrare.

Solutio.

Ad analogiam praecedentis solutionis substituere licet pro x, y, p functiones variabilis z et aliarum trium quantitatum a, b, c. Sic aequatio transformatur in aliam inter z, a, b, c. Jam istae functiones hac lege sunt definiendae, ut ex aequatione hac transformata exeant z et dz.

Differentialia quantitatum x, y, p, tanquam functionum τῶν z, a, b, c, sequenti ratione exprimantur:

$$dx = Xdz + \chi da + \chi' db + \chi'' dc$$
$$dy = Ydz + \eta da + \eta' db + \eta'' dc$$
$$dp = \Pi dz + \pi da + \pi' db + \pi'' dc.$$

Porro cum sint P, Q, R, functiones datae τῶν z, x, y, p, earum differentialia hanc formam habebunt:

$$dP = P'dx + P''dy + P'''dp + P''''dz$$
$$dQ = Q'dx + Q''dy + Q'''dp + Q''''dz$$
$$dR = R'dx + R''dy + R'''dp + R''''dz,$$

ubi $P', .. P''''$; $Q', .. Q''''$; $R', .. R''''$; itidem sunt functiones datae praedictarum quantitatum. Jam aequatio proposita in hanc abit:

$$o = \begin{array}{r|r|r|r|l} PX & dz + P\chi & da + P\chi' & db + P\chi'' & dc \\ +QY & +Q\eta & +Q\eta' & +Q\eta'' & \\ +R\Pi & +R\pi & +R\pi' & +R\pi'' & \\ -1 & & & & \end{array}$$

Hinc coefficientem τοῦ $dz = o$ posito prodit:

1) $1 = PX + QY + R\Pi$

Restat igitur aequatio:

$$o = da + \frac{P\chi' + Q\eta' + R\pi'}{P\chi + Q\eta + R\pi}\, db + \frac{P\chi'' + Q\eta'' + R\pi''}{P\chi + Q\eta + R\pi}\, dc.$$

Jam ut coefficientes τῶν db, dc, a z liberi fiant, eorundem differentialia secundum z accepta, quantitatibus a, b, c, constantium instar habitis, = o ponenda sunt. Hinc uti §. praecedente debet esse

$$\frac{d^z(P\chi + Q\eta + R\pi)}{P\chi + Q\eta + R\pi} = \frac{d^z(P\chi' + Q\eta' + R\pi')}{P\chi' + Q\eta' + R\pi'} = \frac{d^z(P\chi'' + Q\eta'' + R\pi'')}{P\chi'' + Q\eta'' + R\pi''}.$$

Est autem $d^z(P\chi + Q\eta + R\pi) = \begin{cases} P d^z\chi + Q d^z\eta + R d^z\pi \\ + \chi d^z P + \eta d^z Q + \pi d^z R \end{cases}$

At vero ex nota differentiationis functionum lege est

$d^z\eta = d^z Y$, $d^z\chi = d^z X$, $d^z\pi = d^z \Pi$, hinc

$$P d^z\chi + Q d^z\eta + R d^z\pi = P d^z X + Q d^z Y + R d^z \Pi =$$
$$d^z(PX + QY + R\Pi) - X d^z P - Y d^z Q - \Pi d^z R = - X d^z P - Y d^z Q - \Pi d^z R,$$

ob $PX + QY + R\Pi = 1$. Inde fit

$$d^z(P\chi + Q\eta + R\pi) = - X d^z P - Y d^z R - \Pi d^z R + \chi d^z P + \eta d^z Q + \pi d^z R.$$

Est porro $d^z P = P' d^z x + P'' d^z y + P''' d^z p + P'''' d^z z = P'\chi + P''\eta + P'''\pi$; simili modo $d^z Q = Q'\chi + Q''\eta + Q'''\pi$; $d^z R = R'\chi + R''\eta + R'''\pi$; praeterea est $d^z P = P' d^z x + P'' d^z y + P''' d^z p + P'''' =$ $P'X + P''Y + P'''\pi + P''''$; et $d^z Q = Q'X + Q''Y + Q'''\pi + Q''''$, $d^z R = R'X + R''Y + R'''\Pi + R''''$.

Quibus substitutis prodit $d^z(P\chi + Q\eta + R\pi) =$

$$\begin{array}{r|r|r|l} P'X & \chi + Q'X & \eta + R'X & \pi, \\ + P''Y & + Q''Y & + R''Y & \\ + P'''\Pi & + Q'''\Pi & + R'''\Pi & \\ + P'''' & + Q'''' & + R'''' & \\ - XP' & - XP'' & - XP''' & \\ - YQ' & - YQ'' & - YQ''' & \\ - \Pi R' & - \Pi R'' & - \Pi R''' & \end{array}$$

et $\dfrac{d^z(P\chi + Q\eta + R\pi)}{P\chi + Q\eta + R\pi} =$

$$\frac{\begin{array}{r|r|r|l} (P'' - Q')Y & \chi + (Q' - P'')X & \eta + (R' - P''')X & \pi \\ + (P''' - R')\Pi & + (Q''' - R'')\Pi & + (R'' - Q''')Y & \\ + P'''' & + Q'''' & + R'''' & \end{array}}{P\chi + Q\eta + R\pi.}$$

Hinc

Hinc permutando χ, η, π cum χ', η', π'; χ'', η'', π'', prodeunt reliqui duo quotientes $\frac{d^z(P\chi' + Q\eta' + R\pi')}{P\chi' + Q\eta' + R\pi'}$, $\frac{d^z(P\chi'' + Q\eta'' + R\pi'')}{P\chi'' + Q\eta'' + R\pi''}$.

Qui tres quotientes invicem erunt aequales, quippe a χ, η, π independentes, si ponatur 2) $Q\,[(P'' - Q')\,Y + (P''' - R')\,\Pi + P'''']$
$$= P\,[(Q' - P'')\,X + (Q''' - R'')\,\Pi + Q'''']$$
3) $R\,[(P'' - Q')\,Y + (P''' - R')\,\Pi + P'''']$
$$= P\,[(R' - P''')\,X + (R'' - Q''')\,Y + R''''].$$

Inde tres quantitates incognitae X, Y, Π his tribus aequationibus definiuntur:

1) $1 = PX + QY + R\Pi$

2) $PQ'''' - QP'''' + P(Q' - P'')X - Q(P'' - Q')Y + \begin{pmatrix} PQ''' - PR'' \\ -QP''' + QR' \end{pmatrix}\Pi = 0$

3) $PR'''' - RP'''' + P(R' - P''')X + \begin{pmatrix} PR'' - PQ''' \\ -RP'' + RQ' \end{pmatrix}Y - R(P''' - R')\Pi = 0$

Sumendo ex (1) $PX = 1 - QY - R\Pi$, et substituendo in (2), (3), prodit

1) $\left\{\begin{matrix} PQ'''' - QP'''' + \\ + Q' - P'' \end{matrix}\right. \begin{pmatrix} PQ''' + QR' + RP'' \\ -QP''' - RQ' - PR'' \end{pmatrix}\Pi = 0$

2) $\left\{\begin{matrix} PR'''' - RP'''' + \\ + R' - P''' \end{matrix}\right. \begin{pmatrix} PR'' + Q'R + P'''Q \\ -RP'' - R'Q - PQ''' \end{pmatrix}Y = 0.$

Hinc tandem sequitur

$$\Pi = \frac{QP'''' - PQ'''' + P'' - Q'}{PQ''' - QP''' + QR' - RQ' + RP'' - PR''},$$

$$Y = \frac{PR'''' - RP'''' + R' - P'''}{PQ''' - P'''Q + QR' - Q'R + RP'' - PR''}.$$

Quibus valoribus substitutis, calculis rite subductis, fit

$$X = \frac{1 - QY - R\Pi}{P} = \frac{RQ'''' - QR'''' + Q''' - R''}{PQ''' - P'''Q + QR' - RQ' + RP'' - PR''}.$$

Sic itaque X, Y, Π per functiones datas τῶν x, y, z, p exprimuntur.

Quodsi jam, uti in praecedenti problemate (§. 5.), in formulis assumtis pro dx, dy, dp, quantitates a, b, c tanquam constantes considerentur, ex aequationibus

$$dx = Xdz$$
$$dy = Ydz$$
$$dp = \Pi dz$$

x, y, p per z et tres constantes arbitrarias a, b, c integrando exprimere licet (§. 2.); earundemque deinde expressionum differentialia completa, dum etiam a, b, c variant, sponte in formas assumtas pro dx, dy, dp, abeunt. Sic igitur tres functiones desideratae τῶν z, a, b, c inventae sunt, quas pro x, y, q substituendo, aequatio differentialis proposita transformatur in novam aequationem inter tres tantum variabiles a, b, c. Quo autem pacto haec aequatio per systema duarum aequationum integrari possit, in praecedenti solutione (§. 3.) satis declaratum est.

Ceterum cum rursus, sicuti x, z, p per y, a, b, c dantur, ita vice versa a, b, c pro functionibus datis τῶν x, y, z, p haberi possint, sponte apparet, integrationem aequationis propositae $dz = Pdx + Qdy + Rdp$ per duas aequationes hujus formae exprimi:

$$1)\ F(x, y, p, z) = \psi\,[f(x, y, p, z)]$$
$$2)\ \overset{1}{F}(x, y, p, z) = \psi'\,[f(x, y, p, z)]$$

ubi signa functionalia significatu supra explicato accipienda sunt.

§. 5.

Problema III.

Aequationem differentialem vulgarem primi ordinis inter quinque variabiles per systema trium aequationum integrare.

Solutio.

Sit aequatio proposita inter quinque variabiles u, x, y, z, p, haec:

$$du = Pdx + Qdy + Rdz + Sdp,$$

existentibus P, Q, R, S functionibus datis τῶν u, x, y, z, p. Fingamus, p esse constantem, tum aequatio abit in hanc: $du = Pdx + Qdy + Rdz$; quam tanquam aequationem inter quatuor variabiles[g]) per systema duarum aequationum ex praecedenti problemate (§. 4.) integrare licet. Quae integratio praebet:

g) Haec aequatio non tantum est quatuor terminorum, verum etiam consideranda est tanquam aequatio inter quatuor variabiles. Ceterum in omni hac disquisitione aequationes n terminorum, et n variabilium invicem distinguendae sunt.

$$1)\ F(x, y, z, u) = \psi\,[f(x, y, z, u)]$$

$$2)\ \overset{1}{F}(x, y, z, u) = \psi'\,[f(x, y, z, u)].$$

At cum in hac integratione supponatur quantitas p constans, eademque in coefficientibus P, Q, R occurrat, haec quantitas etiam in expressiones ab illis coefficientibus pendentes, signis F, $\overset{1}{F}$, f denotatas, praeter x, y, z, u determinato modo ingredietur, sicque loco $F(x, y, z, u)$ poni debet $F(x, y, z, u, p)$, idemque de functionibus per signa $\overset{1}{F}$ et f notatis valet, quae erunt expressiones data ratione quinque quantitates x, y, z, u, p involventes. Porro cum functio signo ψ designata sit functio arbitraria, ea utcunque etiam quantitatem p involvere potest. Etenim functio $f(x, y, z, u, p)$ tanquam expressio analytica ex quantitatibus x, y, z, u, p dato modo composita, brevitatis gratia exprimatur una littera f, tum functionis arbitrariae ψf hanc formam fingere licet: $\psi f = \mathfrak{A} f^{\alpha} + \mathfrak{B} f^{\beta} + \mathfrak{C} f^{\gamma} + \ldots$ ubi coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. quaecunque constantes sunt, hincque praeter numeros absolutos sive revera constantes etiam quantitate ficta constante p quocunque modo affectae esse possunt. Sic autem expressio haec pro ψf nil aliud est quam forma generalis functionis duarum quantitatum f et p, indeque loco ψf poni debet $\psi(f, p)$. Quare praedictae duae aequationes casu a nobis supposito in has abeunt:

$$1)\ F(x, y, z, u, p) = \psi\,[f(x, y, z, u, p), p]$$

$$2)\ \overset{1}{F}(x, y, z, u, p) = \psi'\,[f(x, y, z, u, p)]$$

ubi signum ψ' secundum notationem La Grangianam supra §. 3. memoratam accipiendum est.

Quae binae aequationes necessariae quidem sunt ad integrationem completam aequationis differentialis propositae, cum haec sine ulla limitatione, hincque etiam pro constante p, valeat: at eaedem non solae sufficiunt, cum aequatio differentialis non tantum pro constante p obtinere debeat. Ad inveniendam tertiam aequationem, qua cum illis combinata integrale completum exhibeatur, prima aequatio differentianda est, ita ut etiam p instar variabilis tractetur. Tum fit, secundum notationem modo laudatam,

$$F'x\,.\,dx + F'y\,.\,dy + F'z\,.\,dz + F'u\,.\,du + F'p\,.\,dp = \psi'f\,.\,df + \psi'p\,.\,dp,$$

litteram f brevitatis gratia, uti antea dictum, adhibendo: inde ob

$$df = f'x\,.\,dx + f'y\,.\,dy + f'z\,.\,dz + f'u\,.\,du + f'p\,.\,dp,$$

et substituendo $\psi' f = \overset{1}{F}(x, y, z, u, p)$ ex aequatione (2), fit

$$\begin{cases} F'x \, . \, dx + F'y \, . \, dy + F'z \, . \, dz + F'u \, . \, du + F'p \, . \, dp = \\ \overset{1}{F}(x, y, z, u, p) \, . \, (f x \, . \, dx + f y \, . \, dy + f z \, . \, dz + f u \, . \, du + f' p \, . \, dp) + \psi' p \, . \, dp \end{cases}$$

vel $du = \mathfrak{P} dx + \mathfrak{Q} dy + \mathfrak{R} dz + \mathfrak{S} dp$, ubi

$$\mathfrak{P} = \frac{F'x - \overset{1}{F}(x, y, z, u, p) \, . \, f x}{f u \, . \, \overset{1}{F}(x, y, z, u, p) - F'u},$$

$$\mathfrak{Q} = \frac{F'y - \overset{1}{F}(x, y, z, u, p) \, . \, f y}{f u \, . \, \overset{1}{F}(x, y, z, u, p) - F'u},$$

$$\mathfrak{R} = \frac{F'z - \overset{1}{F}(x, y, z, u, p) \, . \, f' z}{f' u \, . \, \overset{1}{F}(x, y, z, u, p) - F'u},$$

$$\mathfrak{S} = \frac{F'p - \overset{1}{F}(x, y, z, u, p) \, . \, f' p - \psi' p}{f u \, . \, \overset{1}{F}(x, y, z, u, p) - F'u}.$$

Functiones litteris F', f' denotatas itidem pro functionibus datis τῶν x, y, z, u, p, habendas esse, in aperto est. Jam vi integrationis ope praecedentis problematis inventae, posito p constante vel $dp = 0$, aequatio $du = \mathfrak{P} dx + \mathfrak{Q} dy + \mathfrak{R} dz$ consentire debet cum aequatione proposita $du = P dx + Q dy + R dz$. Inde aequationes $\mathfrak{P} = P$, $\mathfrak{Q} = Q$, $\mathfrak{R} = R$, identicae esse et pro quovis valore constantis arbitrariae p valere debent: ubi nunc perinde est, sive haec quantitas indeterminata tanquam variabilis, sive tanquam constans consideretur. Quare ut aequatio $du = \mathfrak{P} dx + \mathfrak{Q} dy + \mathfrak{R} dz + \mathfrak{S} dp$ cum aequatione proposita $du = P dx + Q dy + R dz + S dp$ ex omni parte conspiret, nil aliud requiritur, quam ut sit insuper $\mathfrak{S} = S$: unde prodit $F'p - \overset{1}{F}(x, y, z, u, p) \, . \, f p - \psi' p = S \, . \, [f u \, . \, \overset{1}{F}(x, y, z, u, p) - F'u]$, vel $\psi' p = F'p - \overset{1}{F}(x, y, z, u, p) \, . \, f p - S \, . \, f u \, . \, \overset{1}{F}(x, y, z, u, p) + S \, . \, F'u$.

Sic igitur etiam $\psi' p$ aequalis reperitur functioni datae τῶν x, y, z, u, p, quam littera $\overset{2}{F}$ notemus. Quare tandem integratio completa aequationis propositae systemate harum trium aequationum comprehenditur:

1) $F\ (x, y, z, u, p) = \psi\ [f(x, y, z, u, p), p]$

2) $\overset{1}{F}\ (x, y, z, u, p) = \psi\ [f(x, y, z, u, p)]$

3) $\overset{2}{F}\ (x, y, z, u, p) = \psi\ p.$

§. 6.

Problema IV.

Aequationem differentiarum partialium inter quatuor variabiles u, z, x, y, complete integrare.

Solutio.

Sit $du = p\,dx + q\,dy + r\,dz$, tum data supponitur relatio inter tres quotientes differentiales $p = \frac{du}{dx}$, $q = \frac{du}{dy}$, $r = \frac{du}{dz}$, et quatuor variabiles u, x, y, z, ex qua quaeritur relatio inter has ipsas quatuor quantitates. Jam aequationem praedictam considerare licet tanquam aequationem inter sex variabiles u, x, y, z, p, q, per quas ipsas etiam r datur. Quam aequationem differentialem per systema trium aequationum finitarum integrare oportet, ex quibus deinceps eliminando p, q sponte prodit aequatio quaesita inter u, x, y, z. Quum vero in praecedenti problemate (§. 5.) ostensum sit, quomodo aequatio differentialis vulgaris inter quinque variabiles per systema trium aequationum integrari queat, nil aliud nunc requiritur, quam ut aequatio proposita differentiarum partialium in aequationem differentialem vulgarem inter quinque variabiles transformetur. Quem in finem ponamus, uti supra §. 3. 4, pro u, x, y, p, q, substitui functiones quantitatis z, aliarunque quinque quantitatum a, b, c, e, f, quarum functionum differentialia hanc formam habebunt:

$$dx = X\,dz + \chi\,da + \chi'\,db + \chi''\,dc + \chi'''\,de + \chi''''\,df$$
$$dy = Y\,dz + \eta\,da + \eta'\,db + \eta''\,dc + \eta'''\,de + \eta''''\,df$$
$$du = U\,dz + \upsilon\,da + \upsilon'\,db + \upsilon''\,dc + \upsilon'''\,de + \upsilon''''\,df$$
$$dp = P\,dz + \pi\,da + \pi'\,db + \pi''\,dc + \pi'''\,de + \pi''''\,df$$
$$dq = Q\,dz + q\,da + q'\,db + q''\,dc + q'''\,de + q''''\,df.$$

Cum porro ex relatione data p exprimere liceat per x, y, z, u, p, q, ejus differentiale hanc formam induet:

$$dr = r'\,dx + r''\,dy + r'''\,dz + r^{IV}\,du + r^{V}\,dp + r^{VI}\,dq,$$

ubi r', r'', ... r^{VI} sunt functiones datae illarum sex quantitatum. Quibus praemissis aequatio proposita $du = p\,dx + q\,dy + r\,dz$ in hanc abit:

$$
\begin{array}{r|r|r|r|r|r|l}
o = pX & dz + p\chi & da + p\chi' & db + p\chi'' & dc + p\chi''' & de + p\chi^{IV} & df. \\
+ qY & + q\eta & + q\eta' & + q\eta'' & + q\eta''' & + q\eta^{IV} & \\
+ r & - \upsilon & - \upsilon' & - \upsilon'' & - \upsilon''' & - \upsilon^{IV} & \\
- U & & & & & &
\end{array}
$$

Quo nunc ex hac aequatione tam dz quam z exeant, ponendum est

$$1)\ U = pX + qY + r;$$

deinde esse debet $\dfrac{d^z(p\chi + qy - \upsilon)}{p\chi + qy - \upsilon} = \dfrac{d^z(p\chi' + qy' - \upsilon')}{p\chi' + qy' - \upsilon'} = \dfrac{d^z(p\chi'' + qy'' - \upsilon'')}{p\chi'' + qy'' - \upsilon''}$

$$= \frac{d^z(p\chi''' + qy''' - \upsilon''')}{p\chi''' + py''' - \upsilon'''} = \frac{d^z(p\chi^{IV} + qy^{IV} - \upsilon^{IV})}{p\chi^{IV} + qy^{IV} - \upsilon^{IV}}.$$

Evolvere igitur oportet $d^z(p\chi + qy - \upsilon)$, ita ut in differentiatione sola z ceu variabilis tractetur, a, b, c, e, f pro constantibus habitis. Est autem $d^z(p\chi + qy - \upsilon) = p\,d^z\chi + q\,d^z y - d^z\upsilon + \chi\,d^z p + y\,d^z q$

Pars prior $p\,d^z\chi + q\,d^z y - d^z\upsilon$ est

$= p\,d^zX + q\,d^zY - d^zU = d^z(pX + qY - U) - X\,d^z p - Y\,d^z q$

$= -d^z\upsilon - X\,d^z p - Y\,d^z q$, vi aequationis (1). At vero est $d^z p = \pi$, $d^z q = q$, $d^z r = r'd^z x + r''d^z y + r^{IV}d^z u + r^V d^z p + r^{VI} d\,q$ $= r'\chi + r''\eta + r^{IV}\upsilon + r^V p + r^{VI} q$; $d^z p = P$, $d^z q = Q$. Hinc fit

$d^z(p\chi + q\eta - \upsilon)$

$$= \begin{array}{r|r|r|r|l} - r' & \chi - r'' & \eta - r^{IV}.\,\upsilon - r^V & \pi - r^{VI} & q, \\ + P & + Q & - X & - Y & \end{array}$$

et

$$\frac{d^z(p\chi + q\eta - \upsilon}{p\chi + q\eta - \upsilon} = \frac{\begin{array}{r|r|r|r|l} r^{IV}.\upsilon + r' & \chi + r'' & \eta + r^V & \pi + r^{VI} & q \\ - P & - Q & + X & + Y & \end{array}}{\upsilon - p\chi - q\eta}.$$

Permutando χ, η, υ, cum $\chi'\ \eta'\ \upsilon'$; χ'', η'', υ''; χ''', η''', υ'''; χ^{IV}, η^{IV}, υ^{IV}, reliqui quatuor quotientes prodeunt. Qui omnes erunt inter se et primo aequales, quippe a υ, χ, η independentes, si ponatur

2) $r^V + X = o$

3) $r^{VI} + Y = o$

4) $r' - P = - p\,r^{IV}$

5) $r'' - Q = - q\,r^{IV}$

Hinc prodit 1) $X = - r^V$;

2) $Y = - r^{VI}$

$$3)\ P = r' + p r^{IV}$$

$$4)\ Q = r'' + q r^{IV}$$

Unde tandem sequitur $5)\ U = pX + qY + r$

$$= r - p r^{V} - q r^{VI}$$

Sic igitur quantitates X, Y, U, P, Q pro functionibus datis τῶν x, y, z, u, p, q haberi possunt. Quod si nunc in formulis differentialibus assumtis a, b, c, e, f, constantium instar tractentur, ex aequationibus differentialibus auxiliaribus:

$$1)\ dx = - r^{V} dz$$

$$2)\ dy = - r^{VI} dz$$

$$3)\ du = (r - p r^{V} - q r^{VI}) dz$$

$$4)\ dp = (r' + p r^{IV}) dz$$

$$5)\ dq = (r'' + p r^{IV}) dz$$

ope integrationis (§. 2.) x, y, u, p, q, per z et quinque quantitates constantes arbitrarias a, b, c, e, f exprimere licet: quarum deinceps expressionum differentialia completa, ipsas has quantitates pro variabilibus habendo, formas assumtas sponte recipient. Quos itaque valores pro x, y, u, p, q in aequatione proposita $du = p\,dx + q\,dy + r\,dz$ substituendo, ea abibit in aequationem, quae, exclusa z, quinque tantum quantitates a, b, c, e, f, earumque differentialia continebit. Hujus autem aequationis transformatae integratio ex problemate praecedente (§. 5.) his tribus aequationibus comprehenditur:

$$1)\ F\ (a, b, c, e, f) = \psi\ [f(a, b, c, e\ f), f]$$

$$2)\ \overset{1}{F}\ (a, b, c, e, f) = \psi'\ [f(a, b, c, e, f)]$$

$$3)\ \overset{2}{F}\ (a, b, c, e, f) = \psi f.$$

Jam vero, sicuti x, y, u, p, q dato modo a z, a, b, c, e, f pendent, ita vice versa quantitates a, b, c, e, f, per z, x, y, u, p, q expressas esse concipere licet. Quare functiones $F, \overset{1}{F}, \overset{2}{F}$, et f ceu functiones datae τῶν z, x, y, u, p, q considerandae sunt, nec non ipsa quantitas f talis erit functio, quam signo functionali $\overset{1}{f}$ notemus. Sic igitur integratio completa aequationis propositae differentiarum partialium inter quatuor variabiles systemate trium aequationum hujus formae exhibebitur:

$$1)\ F\ (x, y, z, u, p, q) = \psi\ [f(x, y, z, u, p, q), \overset{1}{f}(x, y, z, u, p, q)]$$

$$2)\ \overset{1}{F}(x, y, z, u, p, q) = \psi'\,[f(x, y, z, u, p, q)]$$

$$3)\ \overset{2}{F}(x, y, z, u, p, q) = \psi'\,[\overset{1}{f}(x, y, z, u, p, q)],$$

ex quibus, si concipiantur eliminatae quantitates p, q, prodit aequatio quaesita inter x, y, z, u. Quae porro integratio, cum functionem arbitrariam duarum quantitatum complectatur, pro completa est habenda.

§. 7.

Problema V.

Aequationem differentialem vulgarem inter sex variabiles per systema trium aequationum finitarum integrare.

Solutio.

Cum aequatio differentialis inter quinque variabiles ex problemate tertio per systema trium aequationum integrabilis sit, ostendendum est, aequationem differentialem inter sex variabiles in aliam transformari posse, quae quinque tantum variabiles earumque differentialia comprehendat. Sit aequatio proposita inter sex variabiles u, x, y, z, p, q haec:

$du = P dx + Q dy + R dz + S dp + T dq$, ubi P, Q, R, S, T sunt functiones datae earundem variabilium. Quare earum differentialia sic exprimentur:

$$dP = P' dx + P'' dy + P''' dz + P^{IV} dp + P^{V} dq + P^{VI} du$$
$$dQ = Q' dx + Q'' dy + Q''' dz + Q^{IV} dp + Q^{V} dq + Q^{VI} du$$
$$dR = R' dx + R'' dy + R''' dz + R^{IV} dp + R^{V} dq + R^{VI} du$$
$$dS = S' dx + S'' dy + S''' dz + S^{IV} dp + S^{V} dq + S^{VI} du$$
$$dT = T' dx + T'' dy + T''' dz + T^{IV} dp + T^{V} dq + T^{VI} du$$

existentibus $P', \ldots P^{VI}$; $Q' \ldots Q^{VI}$; .... $T', \ldots T^{VI}$, itidem functionibus datis istarum variabilium. Jam ponamus, loco x, y, z, p, q, substitui functiones quantitatis u, et quinque novorum quantitatum a, b, c, e, f: tum illarum differentialia ita exprimere licet:

$$dx = X du + \chi da + \chi' db + \chi'' dc + \chi''' de + \chi'''' df$$
$$dy = Y du + \eta da + \eta' db + \eta'' dc + \eta''' de + \eta'''' df$$
$$dz = Z du + \zeta da + \zeta' db + \zeta'' dc + \zeta''' de + \zeta'''' df$$
$$dp = P du + \pi da + \pi' db + \pi'' dc + \pi''' de + \pi'''' df$$
$$dq = Q du + q da + q' db + q'' dc + q''' de + q'''' df$$

Quae substituendo aequatio proposita in hanc abit:

o =

$$
0 = \begin{array}{r|r|r|r|l}
PX & du + P\chi & da + P\chi' & db + \ldots + P\chi^{iv} & df \\
+ QY & + Q\eta & + Q\eta' & + Q\eta^{iv} & \\
+ RZ & + R\zeta & + R\zeta' & + R\zeta^{iv} & \\
+ S\mathfrak{P} & + S\pi & + S\pi' & + S\pi^{iv} & \\
+ T\Omega & + Tq & + Tq' & + Tq^{iv} & \\
- 1 & & & &
\end{array}
$$

Quo jam haec aequatio a du libera fiat, ponendum est

$$1)\ PX + QY + RZ + S\mathfrak{P} + T\Omega = 1.$$

Deinde ut eadem etiam ab u liberetur, debet esse, uti supra (§. 6.)

$$\frac{d^n(P\chi + Q\eta + R\zeta + S\pi + Tq)}{P\chi + Q\eta + R\zeta + S\pi + Tq}$$

$$= \frac{d^n(P\chi' + Q\eta' + R\zeta' + S\pi' + Tq')}{P\chi' + Q\eta' + R\zeta' + S\pi' + Tq')}$$

. . . .

. . . .

$$= \frac{d^n(P\chi^{iv} + Q\eta^{iv} + R\zeta^{iv} + S\pi^{iv} + Tq^{iv})}{P\chi^{iv} + Q\eta^{iv} + R\zeta^{iv} + S\pi^{iv} + Tq^{iv}}$$

Quare methodo hactenus adhibita evolvendae sunt differentialia in numeratoribus harum quinque fractionum occurrentia. Est autem

$$d^n(P\chi + Q\eta + R\zeta + S\pi + Tq)$$

$$= \begin{cases} \chi d^n P + \eta d^n Q + \zeta d^n R + \pi d^n S + q d^n T \\ + P d^n \chi + Q d^n \eta + R d^n \zeta + S d^n \pi + T d^n q \end{cases}$$

$$= \begin{cases} \chi d^n P + \eta d^n Q + \zeta d^n R + \pi d^n S + q d^n T \\ + P d^n X + Q d^n Y + R d^n Z + S d^n \mathfrak{P} + T d^n \Omega \end{cases}$$

$$= \begin{cases} \chi d^n P + \eta d^n Q + \zeta d^n R + \pi d^n S + q d^n T \\ + d^n (PX + QY + RZ + S\mathfrak{P} + T\Omega) \\ - X d^n P - Y d^n Q - Z d^n R - \mathfrak{P} d^n S - \Omega d^n T \end{cases}$$

vel ex aequatione (1),

$$= \begin{cases} \chi d^n P + \eta d^n Q + \zeta d^n R + \pi d^n S + q d^n T \\ - X d^n P - Y d^n Q - Z d^n R - \mathfrak{P} d^n S - \Omega d^n T. \end{cases}$$

At vero est $d^n P = P' d^n x + P'' d^n y + P''' d^n z + P^{iv} d^n p + P^{v} d^n q$

$$= P'\chi + P''\eta + P'''\zeta + P^{iv}\pi + P^{v} q,$$

$$d^u P = P' d^u x + P'' d^u y + P''' d^u z + P^{iv} d^u p + P^{v} d^u q + P^{vi}$$
$$= P'X + P''Y + P'''Z + P^{iv}\mathfrak{P} + P^{v}\mathfrak{Q} + P^{vi}$$

Simili modo expressis differentialibus τῶν Q, R, S, T, secundum a et u, fit

$$d^u (P\chi + Q\eta + R\zeta + S\pi + Tq)$$

$$\begin{array}{rl|rl|rl|rl|rl|l}
= & P'X & \chi + & Q'X & \eta + & R'X & \zeta + & S'X & \pi + & T'X & q \\
 & +P''Y & & +Q''Y & & +R''Y & & +S''Y & & +T''Y & \\
 & +P'''Z & & +Q'''Z & & +R'''Z & & +S'''Z & & +T'''Z & \\
 & +P^{iv}\mathfrak{P} & & +Q^{iv}\mathfrak{P} & & +R^{iv}\mathfrak{P} & & +S^{v}\mathfrak{P} & & +T^{iv}\mathfrak{P} & \\
 & +P^{v}\mathfrak{Q} & & +Q^{v}\mathfrak{Q} & & +R^{v}\mathfrak{Q} & & +S^{v}\mathfrak{Q} & & +T^{v}\mathfrak{Q} & \\
 & +P^{vi} & & +Q^{vi} & & +R^{vi} & & +S^{vi} & & +T^{vi} & \\
 & -XP' & & -XP'' & & -XP''' & & -XP^{iv} & & -XP^{v} & \\
 & -YQ' & & -YQ'' & & -YQ''' & & -YQ^{iv} & & -YQ^{v} & \\
 & -ZR' & & -ZR'' & & -ZR''' & & -ZR^{iv} & & -ZR^{v} & \\
 & -\mathfrak{P}S' & & -\mathfrak{P}S'' & & -\mathfrak{P}S''' & & -\mathfrak{P}S^{iv} & & -\mathfrak{P}S^{v} & \\
 & -\mathfrak{Q}T' & & -\mathfrak{Q}T'' & & -\mathfrak{Q}T''' & & -\mathfrak{Q}T^{iv} & & -\mathfrak{Q}T^{v} &
\end{array}$$

Jam quotiens $\dfrac{d^u (P\chi + Q\eta + R\zeta + S\pi + Tq)}{P\chi + Q\eta + R\zeta + S\pi + Tq}$

eundem valorem servabit, permutando $\chi, \eta, \zeta, \pi, q$ cum $\chi', \eta', \zeta', \pi', q'$; χ''; $\eta'', \zeta'', \pi'', q''$; . . . $\chi^{iv}, \eta^{iv}, \eta^{iv}, \pi^{iv}, q^{iv}$; si numeratorem ad formam $MP\chi + MQ\eta + MR\zeta + MS\pi + MTq$ revocare licet: quippe tum quoque quotientes abeunt in M. Illud autem locum habebit, si hae supponantur aequationes:

2) $Q\,[P^{vi} + (P'' - Q')\,Y + (P''' - R')\,Z + (P^{iv} - S')\,\mathfrak{P} + (P^{v} - T')\,\mathfrak{Q}]$
$= P\,[Q^{vi} + (Q' - P'')\,X + (Q''' - R'')\,Z + (Q^{iv} - S'')\,\mathfrak{P} + (Q^{v} - T'')\,\mathfrak{Q}]$

3) $R\,[P^{vi} + (P'' - Q')\,Y + (P''' - R')\,Z + (P^{iv} - S')\,\mathfrak{P} + (P^{v} - T')\,\mathfrak{Q}]$
$= P\,[R^{vi} + (R' - P''')\,X + (R'' - Q''')\,Y + (R^{iv} - S''')\,\mathfrak{P} + (R^{v} - T''')\,\mathfrak{Q}]$

4) $S\,[P^{vi} + (P'' - Q')\,Y + (P''' - R')\,Z + (P^{iv} - S')\,\mathfrak{P} + (P^{v} - T')\,\mathfrak{Q}]$
$= P\,[S^{vi} + (S' - P^{iv})\,X + (S'' - Q^{iv})\,Y + (S''' - R^{iv})\,Z + (S^{v} - T^{iv})\,\mathfrak{Q}]$.

5) $T\,[P^{vi} + (P'' - Q')\,Y + (P''' - R')\,Z + (P^{iv} - S')\,\mathfrak{P} + (P^{v} - T')\,Q]$
$= P\,(T^{vi} + (T' - P^{v})\,X + (T'' - Q^{v})\,Y + (T''' - R^{v})\,Z + (T^{iv} - S^{v})\,\mathfrak{P}]$

Ex quibus quatuor aequationibus, junctis cum prima

1) $PX + QY + RZ + S\mathfrak{P} + T\mathfrak{Q} = 1$

quinque quantitates X, Y, Z, $\mathfrak{P}$, $\mathfrak{Q}$, determinare oportet. Calculis rite subductis, prodit $Y = \frac{\mathfrak{M}}{\mathfrak{N}}$, existente

$$\mathfrak{M} = \begin{cases} (PR^{v} - RP^{v} + RT' - TR' + TP''' - PT''')\,(SP^{vi} - PS^{vi} + P^{iv} - S') \\ -(PR^{iv} - RP^{iv} + RS' - SR' + SP''' - PS''')\,(TP^{vi} - PT^{vi} + P^{v} - T'); \\ -(PS^{v} - SP^{v} + ST' - TS' + TP^{iv} - PT^{iv})\,(RP^{vi} - PR^{vi} + P'' - R'); \end{cases}$$

$$\mathfrak{N} = \begin{cases} (PR^{iv} - RP^{iv} + RS' - SR' + SP''' - PS''')\,(PQ^{v} - QP^{v} + QT' - TQ' + TP'' - PT'') \\ -(PR^{v} - RP^{v} + RT' - TR' + TP''' - PT''')\,(PQ^{iv} - QP^{iv} + QS' - SQ' + SP'' - PS'') \\ +(PS^{v} - SP^{v} + ST' - TS' + TP^{iv} - PT^{iv})\,(PQ''' - QP''' + QR' - RQ' + RP'' - PR'') \end{cases}$$

Ex Y prodit Z, permutando invicem litteras Q et R, atque indices $''$ et $'''$; simili modo ex Y prodit $\mathfrak{P}$ vel $\mathfrak{Q}$, permutando in expressione pro Y litteras Q et S, vel Q et T, nec non indices $''$ et ^{iv}, vel $''$ et ^{v}. Sic denominator $S\,2$ invariatus manet, nisi quod signa mutet. Quod ad X attinet, inventis Y, Z, $\mathfrak{P}$, $\mathfrak{Q}$, est $X = \frac{1 - QY - RZ - S\mathfrak{P} - T\mathfrak{Q}}{P}$, vel etiam X prodit ex Y, permutando invicem litteras P et Q, indices $'$ et $''$.

Qua ratione determinatis X, Y, Z, $\mathfrak{P}$, $\mathfrak{Q}$, si fingendo a, b, c, e, f esse constantes, ex aequationibus

$$\begin{aligned} dx &= X\,du \\ dy &= Y\,du \\ dz &= Z\,du \\ dp &= \mathfrak{P}\,du \\ dq &= \mathfrak{Q}\,du \end{aligned}$$

secundum §. 2. exprimantur x, y, z, p, q per u et quinque constantes arbitrarias a, b, c, e, f ex ipsa integratione ingressas, eaedem expressiones, suppositis deinceps a, b, c, e, f, variabilibus, praebebunt functiones τῶν u, a, b, c, e, f, ita comparatas, ut eas pro x, y, z, p, q substituendo aequatio differentialis proposita abeat in aequationem inter quinque quantitates a, b, c, e, f, earumque differentialia. Jam vero ex problemate (3) integratio hujus aequationis transformatae his comprehenditur tribus aequationibus:

1) $F\,(a, b, c, e, f) = \psi\,[f\,(a, b, c, e, f), f]$

2) $\overset{1}{F}\,(a, b, c, e, f) = \psi'\,[f(a, b, c, e, f)]$

3) $\overset{2}{F}\,(a, b, c, e, f) = \psi' f.$

Cum autem singulas quantitates a, b, c, e, f per x, y, z, p, q et u expressas concipere liceat, functiones litteris F, $\overset{1}{F}$, $\overset{2}{F}$, f insignitae, nec non ipsa quantitas f, tanquam functiones datae τῶν x, y, z, p, q, u considerari possunt. Itaque integratio aequationis propositae his tandem absolvitur aequationibus:

$$1)\ F(x, y, z, p, q, u) = \psi\,[f(x, y, z, p, q, u),\ \overset{1}{f}(x, y, z, p, q, u)]$$

$$2)\ \overset{1}{F}(x, y, z, p, q, u) = \psi'\,[f(x, y, z, p, q, u)]$$

$$3)\ \overset{2}{F}(x, y, z\, p, q, u) = \psi'\,[\overset{1}{f}(x, y, z, p, q, u)].$$

§. 8.

Si in praecedente solutione numerator et denominator formulae pro Y evolvuntur, multiplicatione actu instituta, illius termini 72 ad 36, hujus termini 108 ad 60 reducuntur, reliquis se mutuo destruentibus, sicque diviso numeratore et denominatore per communem factorem P, et ponendo $\frac{\mathfrak{M}}{P} = \mathfrak{M}'$, $\frac{\mathfrak{N}}{P} = \mathfrak{N}'$, prodit $Y = \frac{\mathfrak{M}'}{\mathfrak{N}'}$, existente

$$\mathfrak{M}' = \begin{cases} P\,(T'''S^{vi} - T^{vi}S''' + R^{vi}S^{v} - R^{v}S^{vi} + T^{vi}R^{iv} - T^{iv}R^{vi}) \\ + R\,(S'T^{vi} - S^{vi}T' + P^{v}S^{vi} - P^{vi}S^{v} + T^{iv}P^{vi} - T^{vi}P^{iv}) \\ + S\,(R^{v}P^{vi} - R^{vi}P^{v} + T^{vi}P''' - T'''P^{vi} + R^{vi}T' - R'T^{vi}) \\ + T\,(R'S^{vi} - R^{vi}S' + S'''P^{vi} - S^{vi}P''' + R^{vi}P^{iv} - R^{iv}P^{vi}) \\ + R^{v}P^{iv} - R^{iv}P^{v} + R'S^{v} - R^{v}S' + T^{iv}P''' - T'''P^{iv} \\ + S'''P^{v} - S^{v}P''' + S'T''' - S'''T' + R^{iv}T' - R'T^{iv}; \end{cases}$$

porro

$$\mathfrak{N}' = \begin{cases} P\begin{pmatrix} S^{v}Q''' - S'''Q^{v} + S'''T' - S'T''' + R^{iv}Q^{v} - R^{v}Q^{iv} + R'T^{iv} - R^{iv}T' \\ + R^{v}S' - R'S^{v} + T'''Q^{iv} - T^{iv}Q''' \end{pmatrix} \\ + Q\begin{pmatrix} S'T''' - S'''T' + S'''P^{v} - P'''S^{v} + R^{iv}T' - R'T^{iv} + R^{v}P^{iv} - R^{iv}P^{v} \\ + R'S^{v} - R^{v}S' + P'''T^{iv} - T'''P^{iv} \end{pmatrix} \\ + R\begin{pmatrix} S'Q^{v} - S^{v}Q' + T'S'' - S'T'' + P^{v}Q^{iv} - P^{iv}Q^{v} + P^{iv}T' - P'T^{iv} \\ + Q'T^{iv} - T'Q^{iv} + P'S^{v} - P^{v}S' \end{pmatrix} \\ + S\begin{pmatrix} Q'R^{v} - R'Q^{v} + R'T'' - R''T' + P'''Q^{v} - P^{v}Q''' + P'T''' - P'''T' \\ + R''P^{v} - R^{v}P' + Q'''T' - Q'T''' \end{pmatrix} \\ + T\begin{pmatrix} S'''Q' - S'Q''' + S''P''' - S'''P' + R'Q^{iv} - R^{iv}Q' + R^{iv}P' - R'P^{iv} \\ + R''S' - R'S'' + P^{iv}Q''' - P'''Q^{iv} \end{pmatrix}. \end{cases}$$

Ex Y reliquae quatuor quantitates X, Z, 𝔓, 𝔔, modo praedicto (§. 7.) facile deducuntur. Ceterum ex hac transformatione denominator omnibus quinque quantitatibus communis est, quod de denominatore formulae praecedenti §. pro Y inventae quoad X non valet.

§. 9.

Problema VI.

Aequationem differentialem vulgarem inter septem variabiles per systema quatuor aequationum integrare.

Solutio.

Sit proposita aequatio differentialis inter septem variabiles u, x, y, z, t, p, q, haec:

$$du = Pdx + Qdy + Rdz + Sdt + Tdp + Udq,$$

existentibus P, Q, R S, T, U, datis quibuscunque functionibus earundem variabilium. Jam fingendo quantitatem q esse constantem, aequatio abit in aequationem inter sex variabiles, eademque ex problemate praecedente (§. 8.) integrari poterit per systema trium aequationum hujus formae:

$$1)\ F(u,x,y,z\ t,p) = \psi\,[f(u,x,y,z,t,p),\ \overset{1}{f}(u,x,y,z,t,p)]$$

$$2)\ \overset{1}{F}(u,x,y,z,t,p) = \psi'\,[f(u,x,y,z,t,p)]$$

$$3)\ \overset{2}{F}(u,x,y,z,t,p) = \psi'\,[\overset{1}{f}(u,x,y,z,t,p)]$$

Hae autem aequationes adhibitis iisdem ratiociniis, quae supra §. 5. explicata sunt, abeunt in has:

$$1)\ F(u,x,y,z,p,q) = \psi\,[f(u,x,y,z,t,p,q),\ \overset{1}{f}(u,x,y,z,t,p,q),\ q]$$

$$2)\ \overset{1}{F}(u,x,y,z,t,p,q) = \psi'\,[f(u,x,y,z,t,p,q)]$$

$$3)\ \overset{2}{F}(u,x,y,z,t,p,q) = \psi'\,[\overset{1}{f}(u,x,y,z,t,p,q)]$$

Deinde, sumendo primae aequationis differentiale completum, quantitate q etiam instar variabilis tractata, ac substituendo pro $\psi' f$, $\psi' \overset{1}{f}$ expressiones aequationum (2) et (3), tribus illis aequationibus accedit quarta, hujus formae:

$$4)\ \overset{3}{F}(u,x,y,z,t,p,q) = \psi' q.$$

Haec quidem iisdem omnino ratiociniis nituntur, quae supra §. 5. amplius demonstravimus, quaeque repetere superfluum est. Quibus igitur quatuor aequationibus absolvitur integratio aequationis propositae. Signis $F, \overset{1}{F}, \overset{2}{F}, \overset{3}{F}, f, \overset{1}{f}$ functiones datas, signo ψ functionem arbitrariam exprimi, ex superioribus constat (§. 3.).

§. 10.

Problema VII.

Aequationem differentiarum partialium inter quinque variabiles u, x, y, z, t complete integrare.

Solutio.

Sit $du = pdx + qdy + rdz + sdt$, tum datur relatio inter quatuor quotientes differentiales p, q, r, s, et quinque variabiles, ex qua quaeritur relatio inter has ipsas variabiles. Iam aequationem istam considerare licet, tanquam aequationem differentialem vulgarem inter octo variabiles u, x, y, z, t, p, q, r, exclusa quantitate s, quippe per reliquas data. Quae aequatio integranda est per systema quatuor aequationum, ex quibus deinceps eliminando p, q, r, prodit ipsa aequatio quaesita inter u, x, y, z, t. At vero ex praecedenti problemate constat, aequationem differentialem vulgarem inter septem variabiles integrari per systema quatuor aequationum: inde id agitur, ut aequatio nostra proposita transformetur in aequationem differentialem inter septem variabiles.

Quem in finem concipiamus, substitui pro x, y, z, u, p, q, r functiones quantitatis t et septem novarum quantitatum a, b, c, e, f, g, h; sitque

$$dx = Xdt + \chi da + \chi' db + \chi'' dc + \chi''' de + \chi^{iv} df + \chi^{v} dg + \chi^{vi} dh$$
$$dy = Ydt + \eta da + \eta' db \;\ldots\ldots\ldots\ldots\; + \eta^{vi} dh$$
$$dz = Zdt + \zeta da + \zeta' db \;\ldots\ldots\ldots\ldots\; + \zeta^{vi} dh$$
$$du = Udt + \upsilon da + \upsilon' db \;\ldots\ldots\ldots\ldots\; + \upsilon^{vi} dh$$
$$dp = \mathfrak{P} dt + \pi da + \pi' db \;\ldots\ldots\ldots\ldots\; + \pi^{vi} dh$$
$$dq = \mathfrak{Q} dt + q da + q' db \;\ldots\ldots\ldots\ldots\; + q^{vi} dh$$
$$dr = \mathfrak{R} dt + r da + r' db \;\ldots\ldots\ldots\ldots\; + r^{vi} dh$$

Sit porro $ds =$

$$s' dx + s'' dy + s''' dz + s^{iv} dt + s^{v} du + s^{vi} dp + s^{vii} dq + s^{viii} dr,$$

ubi $s', s'', s''', \ldots s^{viii}$, sicuti ipsa s, sunt functiones datae τῶν x, y, z, t, u, p, q, r.

Jam aequatio $du = pdx + qdy + rdz + sdt$ in hanc transmutatur:

$$o = \left.\begin{matrix} pX \\ +qY \\ +rZ \\ +s \\ -U \end{matrix}\right| dt + \left.\begin{matrix} p\chi \\ +q\eta \\ +r\zeta \\ \\ -\upsilon \end{matrix}\right| da + \left.\begin{matrix} p\chi' \\ +q\eta' \\ +r\zeta' \\ \\ -\upsilon' \end{matrix}\right| db + \ldots\ldots + \left.\begin{matrix} p\chi^{vi} \\ +q\eta^{vi} \\ +r\zeta^{vi} \\ \\ -\upsilon^{vi} \end{matrix}\right| dh.$$

Ex hac eliminare oportet dt et t. Quare ponendum est

$$1)\quad U = pX + qY + rZ + s.$$

Deinde quotiens $\dfrac{dt(\upsilon - p\chi - q\eta - r\zeta)}{\upsilon - p\chi - q\eta - r\zeta}$ valorem eundem servare debet, si loco litterarum $\upsilon, \chi, \eta, \zeta$, eaedem indicibus I, II, ... VI, notatae supponantur.

Est autem $d^t(\upsilon - p\chi - q\eta - r\zeta) =$

$$\begin{aligned} & d^t\upsilon - pd^t\chi - qd^t\eta - rd^t\zeta - \chi d^tp - \eta d^tq - \zeta d^tr \\ & = d^aU - pd^aX - qd^aY - rd^aZ - \chi d^tp - \eta d^tq - \zeta d^tr \\ & = \begin{cases} d^a(U - pX - qY - rZ) \\ + Xd^ap + Yd^aq + Zd^ar - \chi d^tp - \eta d^tq - \zeta d^tr \end{cases} \\ & = d^as + Xd^ap + Yd^aq + Zd^ar - \chi d^tp - \eta d^tq - \zeta d^tr. \end{aligned}$$

Est autem $d^ap = \pi$, $d^aq = q$, $d^ar = r$,

$$\begin{aligned} d^as &= s'd^ax + s''d^ay + s'''d^az + s^vd^au + s^{vi}d^ap + s^{vii}d^aq + s^{viii}d^ar \\ &= s'\chi + s''\eta + s'''\zeta + s^v\upsilon + s^{vi}\pi + s^{vii}q + s^{viii}r; \end{aligned}$$

porro $d^tp = \mathfrak{P}$, $d^tq = \mathfrak{Q}$, $d^tr = \mathfrak{R}$.

Hinc fit $\dfrac{d^t(\upsilon - p\chi - q\eta - r\zeta)}{\upsilon - p\chi - q\eta - r\zeta}$

$$= \frac{\left.\begin{matrix} s' \\ -\mathfrak{P} \end{matrix}\right| \chi + \left.\begin{matrix} s'' \\ -\mathfrak{Q} \end{matrix}\right| \eta + \left.\begin{matrix} s''' \\ -\mathfrak{R} \end{matrix}\right| \zeta + s^v.\upsilon + \left.\begin{matrix} s^{vi} \\ +X \end{matrix}\right| \pi + \left.\begin{matrix} s^{vii} \\ +Y \end{matrix}\right| q + \left.\begin{matrix} s^{viii} \\ +Z \end{matrix}\right| r}{\upsilon - p\chi - q\eta - r\zeta}$$

ex qua fractione exeunt litterae $\chi, \eta, \zeta, \upsilon$, nec non π, q, r, si hae assumantur aequationes:

$$\begin{aligned} 2)\quad & s' - \mathfrak{P} = -s^v.p \\ 3)\quad & s'' - \mathfrak{Q} = -s^v.q \\ 4)\quad & s''' = \mathfrak{R} = -s^v.r \end{aligned}$$

$$5)\ s^{vi} + X = 0$$
$$6)\ s^{vii} + Y = 0$$
$$7)\ s^{viii} + Z = 0$$

Hinc fit
$$X = - s^{vi}$$
$$Y = - s^{vii}$$
$$Z = - s^{viii}$$
$$\mathfrak{P} = s' + s^{v} . p$$
$$\mathfrak{Q} = s'' + s^{v} . q$$
$$\mathfrak{R} = s''' + s^{v} . r$$

Tandem prodit ex (1) $U = pX + qY + rZ + s$
$$= s - ps^{vi} - qs^{vii} - rs^{viii}$$

Qua igitur ratione septem quantitates X, Y, Z, $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, U, tanquam functiones datae τῶν x, y, z, u, p, q, r, t, expressae sunt.

Quodsi nunc, suppositis a, b, c, e, f, g, h constantibus, ex aequationibus auxiliaribus

$$dx = Xdt$$
$$dy = Ydt$$
$$dz = Zdt$$
$$du = Udt$$
$$dq = \mathfrak{P}dt$$
$$dq = \mathfrak{Q}dt$$
$$dr = \mathfrak{R}dt$$

quaerantur valores τῶν x, y, z, u, p, q, r per t et septem constantes arbitrarias integratione ingressas a, b, c, e, f, g, h, expressi (§. 2.): tum hic ipsi valores, sumtis eorum differentialibus completis, tractando constantes praedictas tanquam variabiles, conditionem praescriptam adimplebunt, i. e. iisdem pro x, y, z, u, p, q, r, substitutis aequatio proposita

$$du = pdx + qdy + rdz + sdt$$

abitum est in aequationem, quae exclusis t et dt, tantum septem quantitates a, b, c, e, f, g, h, earumque differentialia comprehendet. Jam hujus aequationis transformatae integratio ex praecedenti problemate (§. 9.) his quatuor aequationibus absolvitur:

1) F

$$1)\ F\,(a, b, c, e, f, g, h) = \psi\,[f(a, \ldots h)\ \overset{1}{f}(a, \ldots h), h]$$

$$2)\ \overset{1}{F}\,(a, \ldots h) = \psi'\,[f(a, \ldots h)]$$

$$3)\ \overset{2}{F}\,(a, \ldots h) = \psi'\,[\overset{1}{f}(a, \ldots h)]$$

$$4)\ \overset{3}{F}\,(a, \ldots h) = \psi'\,h.$$

Quod si nunc quantitates a, b, c, e, f, g, h per x, y, z, u, p, q, r et per t expressae concipiantur, hae quatuor aequationes has formas induent:

$$1)\ F\,(x, y, z, t, u, p, q, r) = \psi\,[f(x, \ldots r), \overset{1}{f}(x, \ldots r), \overset{2}{f}(x, \ldots r)]$$

$$2)\ \overset{1}{F}\,(x, \ldots r) = \psi'\,[f(x, \ldots r)]$$

$$3)\ \overset{2}{F}\,(x, \ldots r) = \psi'\,[\overset{1}{f}(x, \ldots r)]$$

$$4)\ \overset{3}{F}\,(x, \ldots r) = \psi'\,(\overset{2}{f}(x, \ldots r))$$

ex quibus, tres quotientes differentiales p, q, r, eliminatos concipiendo, prodit aequatio inter ipsas variabiles x, y, z, t, u; quae est ipsa integratio desiderata, et quidem completa, ob functionem arbitrariam trium quantitatum.

§. 11.

Problema VIII.

Aequationem differentialem vulgarem quamcunque inter octo variabiles per systema quatuor aequationum integrare.

Solutio.

Sit aequatio proposita inter octo variabiles u, x, y, z, t, p, q, r, haec:

$$du = P dx + Q dy + R dz + S dt + T dp + U dq + W dr.$$

Cum P, Q, ... W sint functiones datae τῶν x, y, ... r, ponendum est

$$dP = P' dx + P'' dy + P''' dz + P^{IV} dt + P^{V} dp + P^{VI} dq + P^{VII} dr + P^{VIII} du;$$

et simili modo dQ, dR, ... dW exprimere licet, ubi

$$P', P'', \ldots P^{VIII};\ Q', \ldots Q^{VIII};\ R', \ldots R^{VIII};\ \ldots\ W', \ldots W^{VIII}$$

functiones sunt itidem datae. Jam cum ex problemate (6) §. 9. aequatio differentialis inter septem variabiles per systema quatuor aequationum integrabilis sit, nil aliud requiritur, quam transformatio aequationis propositae in aequationem inter septem variabiles, quae sint a, b, ... h. Quem in finem ponamus, more hactenus servato,

$$dx = X du + \chi da + \chi' db + \ldots + \chi^{VI} dh$$

$$dy = Y du + \eta da + \ldots\ldots$$

$$dz = Z du + \zeta da + \ldots\ldots$$

$$
\begin{aligned}
dt &= \mathfrak{T}du + \tau da + \;.\;.\;.\;.\;.\;.\;. \\
dp &= \mathfrak{P}du + \pi da + \;.\;.\;.\;.\;.\;.\;. \\
dq &= \mathfrak{Q}du + q da + \;.\;.\;.\;.\;.\;.\;. \\
dr &= \mathfrak{R}du + r da + \;.\;.\;.\;.\;.\;.\;.
\end{aligned}
$$

Tum aequatio proposita in hanc abit:

$$
\begin{array}{l|l|l|l|l}
o = PX & du + P\chi & da + P\chi' & db \ldots. + P\chi^{vi} & dh \\
+QY & +Q\eta & +Q\eta' & +Q\eta^{vi} & \\
+RZ & +R\zeta & +R\zeta' & +R\zeta^{vi} & \\
+S\mathfrak{T} & +S\tau & +S\tau' & +S\tau^{vi} & \\
+T\mathfrak{P} & +T\pi & +T\pi' & +T\pi^{vi} & \\
+U\mathfrak{Q} & +Uq & +Uq' & +Uq^{vi} & \\
+W\mathfrak{R} & +Wr & +Wr' & +Wr^{vi} & \\
-1 & & & &
\end{array}
$$

Quae aequatio ut a du et u liberetur, ponendum est

$$1)\ 1 = PX + QY + RZ + S\mathfrak{T} + T\mathfrak{P} + U\mathfrak{Q} + W\mathfrak{R}$$

Deinde quotiens $\dfrac{d^{II}(P\chi + Q\eta + R\zeta + S\tau + T\pi + Uq + Wr)}{P\chi + Q\eta + R\zeta + S\tau + T\pi + Uq + Wr}$

valorem eundem servare debet, ponendo

$$\chi', \eta', \ldots r; \;\ldots.\; \chi^{vi}, \eta^{vi}, \ldots r^{vi} \text{ pro } \chi, \eta, \ldots r.$$

Est autem numerator

$$
\begin{aligned}
&= \begin{cases} Pd^{II}\chi + Qd^{II}\eta + Rd^{II}\zeta + Sd^{II}\tau + Td^{II}\pi + Ud^{II}q + Wd^{II}r \\ +\chi d^{II}P + \eta d^{II}Q + \zeta d^{II}R + \tau d^{II}S + \pi d^{II}T + q d^{II}U + r d^{II}W \end{cases} \\
&= \begin{cases} Pd^{I}X + Qd^{I}Y + Rd^{I}Z + Sd^{I}\mathfrak{T} + Td^{I}\mathfrak{P} + Ud^{I}\mathfrak{Q} + Wd^{I}\mathfrak{R} \\ +\chi d^{II}P + \eta d^{II}Q + \zeta d^{II}R + \tau d^{II}S + \pi d^{II}T + q d^{II}U + r d^{II}W \end{cases} \\
&= \begin{cases} d^{I}(PX + QY + RZ + S\mathfrak{T} + T\mathfrak{P} + U\mathfrak{Q} + W\mathfrak{R}) \\ -Xd^{I}P - Yd^{I}Q - Zd^{I}R - \mathfrak{T}d^{I}S - \mathfrak{P}d^{I}T - \mathfrak{Q}d^{I}U - \mathfrak{R}d^{I}W \\ +\chi d^{II}P + \eta d^{II}Q + \zeta d^{II}R + \tau d^{II}S + \pi d^{II}T + q d^{II}U + r d^{II}W, \end{cases}
\end{aligned}
$$

ubi prima pars ex aequatione (1) evanescit. Et porro

$$
\begin{aligned}
d^{I}P &= P'd^{I}x + P''d^{I}y + P'''d^{I}z + P^{IV}d^{I}t + P^{V}d^{I}p + P^{VI}d^{I}q + P^{VII}d^{I}r \\
&= P\chi + P''\eta + P'''\zeta + P^{IV}\tau + P^{V}\pi + P^{VI}q + P^{VII}r; \\
d^{II}P &= P'd^{II}x + P''d^{II}y + P'''d^{II}z + P^{IV}d^{II}t + P^{V}d^{II}p + P^{VI}d^{II}q + P^{VII}d^{II}r + P^{VIII} \\
&= P'X + P''Y + P'''Z + P^{IV}\mathfrak{T} + P^{V}\mathfrak{P} + P^{VI}\mathfrak{Q} + P^{VII}\mathfrak{R} + P^{VIII};
\end{aligned}
$$

simili modo exprimuntur $d^{a}Q$, $d^{a}R, \ldots d^{a}W$, nec non $d^{u}Q$, $d^{u}R, \ldots d^{u}W$. Hinc numerator praedictus fit

$$
\begin{array}{l|l|l|l|l}
= P'X - XP' & \chi + Q'X - XP'' & \eta + R'X - XP''' & \zeta + S'X - XP^{iv} & \tau \\
+ P''Y - YQ' & + Q''Y - YQ'' & + R''Y - YQ''' & + S''Y - YQ^{iv} & \\
+ P'''Z - ZR' & + Q'''Z - ZR'' & + R'''Z - ZR''' & + S'''Z - ZR^{iv} & \\
+ P^{iv}\mathfrak{T} - \mathfrak{T}S' & + Q^{iv}\mathfrak{T} - \mathfrak{T}S'' & + R^{iv}\mathfrak{T} - \mathfrak{T}S''' & + S^{iv}\mathfrak{T} - \mathfrak{T}S^{iv} & \\
+ P^{v}\mathfrak{P} - \mathfrak{P}T' & + Q^{v}\mathfrak{P} - \mathfrak{P}T'' & + R^{v}\mathfrak{P} - \mathfrak{P}T''' & + S^{v}\mathfrak{P} - \mathfrak{P}T^{iv} & \\
+ P^{vi}\mathfrak{Q} - \mathfrak{Q}U' & + Q^{vi}\mathfrak{Q} - \mathfrak{Q}U'' & + R^{vi}\mathfrak{Q} - \mathfrak{Q}U''' & + S^{vi}\mathfrak{Q} - \mathfrak{Q}U^{iv} & \\
+ P^{vii}\mathfrak{R} - \mathfrak{R}W' & + Q^{vii}\mathfrak{R} - \mathfrak{R}W'' & + R^{vii}\mathfrak{R} - \mathfrak{R}W''' & + S^{vii}\mathfrak{R} - \mathfrak{R}W^{iv} & \\
+ P^{viii} & + Q^{viii} & + R^{viii} & + S^{viii} &
\end{array}
$$

$$
\begin{array}{l|l|l|l}
+ T'X - XP^{v} & \pi + U'X - YP^{vi} & q + W'X - XP^{vii} & r \\
+ T''Y - YQ^{v} & + U''Y - YQ^{vi} & + W''Y - YQ^{vii} & \\
+ T'''Z - ZR^{v} & + U'''Z - ZR^{vi} & + W'''Z - ZR^{vii} & \\
+ T^{iv}\mathfrak{T} - \mathfrak{T}S^{v} & + U^{iv}\mathfrak{T} - \mathfrak{T}S^{vi} & + W^{iv}\mathfrak{T} - \mathfrak{T}S^{vii} & \\
+ T^{v}\mathfrak{P} - \mathfrak{P}T^{v} & + U^{v}\mathfrak{P} - \mathfrak{P}T^{vi} & + W^{v}\mathfrak{P} - \mathfrak{P}T^{vii} & \\
+ T^{vi}\mathfrak{Q} - \mathfrak{Q}U^{v} & + U^{vi}\mathfrak{Q} - \mathfrak{Q}U^{vi} & + W^{vi}\mathfrak{Q} - \mathfrak{Q}U^{vii} & \\
+ T^{vii}\mathfrak{R} - \mathfrak{R}W^{v} & + U^{vii}\mathfrak{R} - \mathfrak{R}W^{vi} & + W^{vii}\mathfrak{R} - \mathfrak{R}W^{vii} & \\
+ T^{viii} & + U^{viii} & + W^{viii} &
\end{array}
$$

Qui numerator si ad formam

$$M\,(P\chi + Q\eta + R\zeta + S\tau + T\pi + Uq + Wr)$$

revocetur, eraditio praedicta adimplebitur. Inde autem haec sequuntur sex aequationes:

$$2)\ o =$$

$$
\left\{
\begin{array}{l}
\begin{array}{l|l|l}
P^{viii}Q - P(Q' - P'')X + Q(P'' - Q')Y + Q(P''' - R') & Z + Q(P^{iv} - S') & \mathfrak{T} \\
- PQ^{viii} \qquad\qquad\qquad\qquad\qquad - P(Q''' - R') & \quad - P(Q^{iv} - S') &
\end{array} \\
\begin{array}{l|l|l|l}
+ Q(P^{v} - T') & \mathfrak{P} + Q(P^{vi} - U') & \mathfrak{Q} + Q(P^{vii} - W') & \mathfrak{R} \\
- P(Q^{v} - T') & \quad - P(Q^{vi} - U') & \quad - P(Q^{vii} - W') &
\end{array}
\end{array}
\right.
$$

$$3)\ o =$$

$$
\left\{
\begin{array}{l}
\begin{array}{l|l|l}
RP^{viii} - P(R' - P''')X + R(P'' - Q') & Y + R(P''' - R')Z + R(P^{iv} - S') & \mathfrak{T} \\
- PR^{viii} \qquad\qquad\qquad - P(R'' - Q''') & \qquad\qquad\qquad - P(R^{iv} - S''') &
\end{array} \\
\begin{array}{l|l|l|l}
+ R(P^{v} - T') & \mathfrak{P} + R(P^{vi} - U') & \mathfrak{Q} + R(P^{vii} - W') & \mathfrak{R} \\
- P(R^{v} - T''') & \quad - P(R^{vi} - U''') & \quad - P(R^{vii} - W''') &
\end{array}
\end{array}
\right.
$$

$$4)\ o =$$

$$\left\{\begin{array}{l|l|l}
SP^{viii} - P(S' - P^{iv})X + S(P'' - Q') & Y + S(P''' - R') & Z + S(P^{iv} - S')\,\mathfrak{T} \\
-PS^{viii} \qquad\qquad - P(S'' - Q^{iv}) & -P(S''' - R^{iv}) & \\
\\
+S(P^{v} - T') & \mathfrak{P} + S(P^{vi} - U') & \mathfrak{Q} + S(P^{vii} - W') \\
-P(S^{v} - T^{iv}) & -P(S^{vi} - U^{iv}) & -P(S^{vii} - W^{iv})
\end{array}\right| \mathfrak{R} \quad :$$

$$5)\ o =$$

$$\left\{\begin{array}{l|l|l}
TP^{viii} - P(T' - P^{v})X + T(P'' - Q') & Y + T(P''' - R') & Z + T(P^{iv} - S') \\
-PT^{viii} \qquad\qquad - P(T'' - Q^{v}) & -P(T''' - R^{v}) & -P(T^{iv} - S^{v})
\end{array}\right| \mathfrak{T}$$

$$\begin{array}{l|l|l}
+T(P^{v} - T')\,\mathfrak{P} + T(P^{vi} - U') & \mathfrak{Q} + T(P^{vii} - W') & \\
\qquad\qquad -P(T^{vi} - U^{v}) & -P(T^{vii} - W^{v}) & \mathfrak{R}
\end{array}$$

$$6)\ o =$$

$$\left\{\begin{array}{l|l|l}
UP^{viii} - P(U' - P^{vi})X + U(P'' - Q') & Y + U(P''' - R') & Z + U(P^{iv} - S') \\
-PU^{viii} \qquad\qquad - P(U'' - Q^{vi}) & -P(U''' - R^{vi}) & -P(U^{iv} - S^{vi})
\end{array}\right| \mathfrak{T}$$

$$\begin{array}{l|l|l}
+U(P^{v} - T') & \mathfrak{P} + U(P^{vi} - U')\,\mathfrak{Q} + U(P^{vii} - W') & \\
-P(U^{v} - T^{vi}) & \qquad\qquad -P(U^{vii} - W^{vi}) & \mathfrak{R}
\end{array}$$

$$7)\ o =$$

$$\left\{\begin{array}{l|l|l}
WP^{viii} - P(W' - P^{vii})X + W(P'' - Q') & Y + W(P''' - R') & Z + W(P^{iv} - S') \\
-PW^{viii} \qquad\qquad - P(W'' - Q^{vii}) & -P(W''' - R^{vii}) & -P(W^{iv} - S^{vii})
\end{array}\right| \mathfrak{T}$$

$$\begin{array}{l|l|l}
+W(P^{v} - T') & \mathfrak{P} + W(P^{vi} - U') & \mathfrak{Q} + W(P^{vii} - W')\,\mathfrak{R} \\
-P\,W^{v} - T^{vii}) & -P(W^{vi} - U^{vii}) &
\end{array}$$

Ex his sex aequationibus, junctis cum prima (1), determinandae sunt septem quantitates X, Y, Z, $\mathfrak{T}$, $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$. Qua determinatione supposita, (quae quidem ex regulis eliminationis vulgaribus calculos admodum longos poscit, de quorum compendiis infra sermo erit), istae quantitates habendae sunt pro functionibus datis octo nostrarum variabilium. Quod si nunc, suppositis a, b, c, . . . h constantibus, ex septem aequationibus auxiliaribus:

$$dx = X du$$
$$dy = Y du$$
$$dz = Z du$$
$$dt = \mathfrak{T} du$$

$$dp = \mathfrak{P}\,du$$
$$dq = \mathfrak{Q}\,du$$
$$dr = \mathfrak{R}\,du$$

variabiles x, y, z, t, p, q, r per u et septem constantes arbitrarias a, b, c, ... h integratione ingressas exprimantur (§. 11.), tum hae ipsae expressiones ita erunt comparatae, ut eas earumque differentialia completa, quantitatibus a, b, ... h, etiam instar variabilium habitis, in aequatione proposita substituendo, haec in aequationem inter septem variabiles a, b, c, ... h transformetur. Jam vero ex supra demonstratis hujus aequationis integratio quatuor aequationibus hujus formae absolvitur:

$$1)\ F(a, b, \ldots h) = \psi\,[f(a, b, \ldots h), \overset{1}{f}(a, b, \ldots h), h]$$
$$2)\ \overset{1}{F}(a, b, \ldots h) = \psi'\,[f(a, b, \ldots h)]$$
$$3)\ \overset{2}{F}(a, b, \ldots h) = \psi'\,[\overset{1}{f}(a, b, \ldots h)].$$
$$4)\ \overset{3}{F}(a, b, \ldots h) = \psi' h.$$

Quod si deinde a, b, ... h per octo variabiles x, y, ... n exprimantur, hae aequationes in has abiturae sunt:

$$1)\ F(x, y, z, t, u, p, q, r) = \psi\,[f(x, \ldots r), \overset{1}{f}(x, \ldots r), \overset{2}{f}(x, \ldots r)]$$
$$2)\ \overset{1}{F}(x, \ldots r) = \psi'\,[f(x, \ldots r)]$$
$$3)\ \overset{2}{F}(x, \ldots r) = \psi'\,[\overset{1}{f}(x, \ldots r)]$$
$$4)\ \overset{3}{F}(x, \ldots r) = \psi'\,[\overset{2}{f}(x, \ldots r)]$$

quarum systemate integratio completa aequationis propositae inter octo variabiles exhibetur.

§. 12.

Problema IX.

Aequationem differentialem vulgarem inter novem variabiles per systema quinque aequationum integrare.

Solutio.

Sit aequatio proposita inter novem variabiles u, x, y, z, t, p, q, r, s, haec:

$$du = P\,dx + Q\,dy + R\,dz + S\,dt + T\,dp + U\,dq + W\,dr + \mathfrak{X}\,ds.$$

Considerando unam harum quantitatum, veluti s, tanquam constantem, aequatio abit in aequationem inter octo variabiles, eademque ex §. praecedente per systema quatuor aequationum integrabilis est. Quae aequationes ex ratiociniis supra §. 5 et 9 adhibitis et explicatis, has formas recipient:

$$1)\ F\,(x, y, z, t, p, q, r, u, s) = \psi\,[\,f(x, y, \ldots s),\ \overset{1}{f}(x, \ldots s),\ \overset{2}{f}(x, \ldots s), s]$$

$$2)\ \overset{1}{F}(x, \ldots s) = \psi'\,[f(x, \ldots s)]$$

$$3)\ \overset{2}{F}(x, \ldots s) = \psi'\,[\overset{1}{f}(x, \ldots s)]$$

$$4)\ \overset{3}{F}(x, \ldots s) = \psi'\,[\overset{2}{f}(x, \ldots s)]$$

Sumatur jam aequationis (1) differentiale completum, habita etiam s variabili, idque differentiale, substitutis pro $\psi' f$, $\psi' \overset{1}{f}$, $\psi' \overset{2}{f}$, valoribus ex aequationibus (2), (3), (4) cognitis, comparetur cum aequatione differentiali proposita, quacum illud identicum esse debet. Quibus rite observatis aequationibus quatuor prioribus accedet aequatio quinta hujus formae:

$$5)\ \overset{4}{F}(x, \ldots s) = \psi' s.$$

Quarum quinque aequationum combinatione absolvitur integratio aequationis propositae.

§. 13.
Problema X.

Aequationem differentiarum partialium inter sex variabiles complete integrare.

Solutio.

Sit $du = p\,dx + q\,dy + r\,dz + s\,dt + w\,dv$, atque detur relatio inter quotientes differentiales p, q, r, s, w, et variabiles u, x, y, z, t, v, ex qua quaeritur relatio inter has ipsas variabiles. Quem ad finem nil aliud requiritur, quam ut aequatio proposita, considerata tanquam aequatio differentialis vulgaris inter decem variabiles, transformetur in aequationem inter novem variabiles, quippe cujus integratio completa per quinque aequationes praecedenti §. inventa est. Ad hanc transformationem obtinendam, in calculum introductis novem quantitatibus notis a, b, c, … h, i, k, more hactenus servato ponamus:

$$dx = X\,dv + \chi\,da + \ldots\ldots + \chi^{viii}\,dk$$
$$dy = Y\,dv + \eta\,da + \ldots\ldots$$

$$dz = Zdv + \zeta da + \ldots\ldots$$
$$dt = \mathfrak{T}dv + \tau da + \ldots\ldots$$
$$du = \mathfrak{U}dv + u da + \ldots\ldots$$
$$dp = \mathfrak{P}dv + \pi da + \ldots\ldots$$
$$dq = \mathfrak{Q}dv + q da + \ldots\ldots$$
$$dr = \mathfrak{R}dv + r da + \ldots\ldots$$
$$ds = \mathfrak{S}dv + s da + \ldots\ldots$$

Sit porro, w considerando tanquam functionem datam variabilium et reliquorum quotientium differentialium,

$$dw = \begin{cases} w'dx + w''dy + w'''dz + w^{iv}dt + w^{v}dv + w^{vi}du + w^{vii}dp \\ + w^{viii}dq + w^{ix}dr + w^{x}ds. \end{cases}$$

Tum aequatio $dv = pdx + qdy + rdz + sdt + wdv$, in hanc abit:

$$o = \left.\begin{matrix} pX \\ +qY \\ +rZ \\ +s\mathfrak{T} \\ +w \\ -\mathfrak{U} \end{matrix}\right| dv + \left.\begin{matrix} p\chi \\ +q\eta \\ +r\zeta \\ +s\tau \\ \\ -u \end{matrix}\right| da + \left.\begin{matrix} p\chi' \\ +q\eta' \\ +r\zeta' \\ +r\tau' \\ \\ -u' \end{matrix}\right| db + \ldots.$$

Quo nunc ex haec aequatione dv et v exeant, ponendum est

$$1)\ pX + qY + rZ + rZ + s\mathfrak{T} + w = \mathfrak{U}$$

Deinde debet esse $\dfrac{d^v(p\chi + q\eta + r\zeta + s\tau - u)}{p\chi + q\eta + r\zeta + s\tau - u}$

$$= \frac{d^v(p\chi' + q\eta' + r\zeta' + s\tau' - u')}{p\chi' + q\eta' + r\zeta' + s\tau' - u'} = \text{etc.} \ldots$$

Est autem $d^v(p\chi + q\eta + r\zeta + s\tau - u)$

$$= pd^v\chi + qd^v\eta + rd^v\zeta + sd^v\tau - d^vu + \chi d^vp + \eta d^vq + \zeta d^vr + \tau d^vs$$
$$= pd^aX + qd^aY + rd^aZ + sd^a\mathfrak{T} - d^a\mathfrak{U} + \chi d^vp + \eta d^vq + \zeta d^vr + \tau d^vs$$
$$= \begin{cases} d^a(pX + qY + rZ + s\mathfrak{T} - \mathfrak{U}) \\ - Xd^ap - Yd^aq - Zd^ar - \mathfrak{T}d^as + \chi d^vp + \eta d^vq + \zeta d^vr + \tau d^vs \end{cases}$$
$$= d^aw - Xd^ap - Yd^aq - Zd^ar - Td^as + \chi d^vp + \eta d^vq + \zeta d^vr + \tau d^vs$$

Est porro $d^ap = \pi$, $d^aq = q$, $d^ar = r$, $d^as = s$;

$$d^aw = \begin{cases} w'd^ax + w''d^ay + w'''d^az + w^{iv}d^at + w^{vi}d^au + w^{vii}d^ap \\ + w^{viii}d^aq + w^{ix}d^ar + w^{x}d^as \end{cases}$$

$$= w'\chi + w''\eta + w'''\zeta + w^{iv}\tau + w^{vi}\upsilon + w^{vii}\pi + w^{viii}\mathfrak{q} + w^{ix}\mathfrak{r} + w^{x}\mathfrak{s};$$

$$d^{v}p = \mathfrak{P},\ d^{v}q = \mathfrak{Q},\ d^{v}r = \mathfrak{R},\ d^{v}s = \mathfrak{S}.$$

Hinc fit $d^{v}(p\chi + q\eta + r\zeta + s\tau - \mathfrak{u})$

$$= \left\{ \begin{array}{l} \begin{array}{c|c|c|c|c|} -w' & \chi - w'' & \eta - w''' & \zeta - w^{iv} & \tau - w^{vi}.\ \mathfrak{u} - w^{vii} \\ +\mathfrak{P} & +\mathfrak{Q} & +\mathfrak{R} & +\mathfrak{S} & -X \end{array}\ \pi \\ \qquad\qquad \begin{array}{c|c|c|} -w^{viii} & \mathfrak{q} - w^{ix} & \mathfrak{r} - w^{x} \\ -Y & -Z & -\mathfrak{T} \end{array}\ \mathfrak{s} \end{array} \right.$$

Quod ponendo $= M(p\chi + q\eta + r\zeta + s\tau - \mathfrak{u})$, hae prodeunt aequationes:

2) $X + w^{vii} = 0$
3) $Y + w^{viii} = 0$
4) $Z + w^{ix} = 0$
5) $\mathfrak{T} + w^{x} = 0$
6) $\mathfrak{P} - w' = p\,w^{vi}$
7) $\mathfrak{Q} - w'' = q\,w^{vi}$
8) $\mathfrak{R} - w''' = r\,w^{vi}$
9) $\mathfrak{S} - w^{iv} = s\,w^{vi}$

Quibus aequationibus junctis cum prima (1) hi prodeunt valores novem quantitatum $X, Y, Z, \mathfrak{T}, \mathfrak{P}, \mathfrak{Q}, \mathfrak{R}, \mathfrak{S}, \mathfrak{U}$,

1) $X = -w^{vii}$
2) $Y = -w^{viii}$
3) $Z = -w^{ix}$
4) $\mathfrak{T} = -w^{x}$
5) $\mathfrak{P} = w' + p\,w^{vi}$
6) $\mathfrak{Q} = w'' + q\,w^{vi}$
7) $\mathfrak{R} = w''' + r\,w^{vi}$
8) $\mathfrak{S} = w^{iv} + s\,w^{vi}$
9) $\mathfrak{U} = w - p\,w^{vii} - q\,w^{viii} - r\,w^{ix} - s\,w^{x}$.

Jam ex aequationibus auxiliaribus

$$dx = X\,dv$$
$$dy = Y\,dv$$
$$dz = Z\,dv$$
$$dt = \mathfrak{T}\,dv$$
$$du = \mathfrak{U}\,dv$$

dp

$$dp = \mathfrak{P}dv$$
$$dq = \mathfrak{Q}dv$$
$$dr = \mathfrak{R}dv$$
$$ds = \mathfrak{S}dv$$

definiendi sunt valores τῶν x, y, z, t, u, p, q, r, s per v et novem constantes arbitrarias a, b, ... k expressi. Quas deinceps expressiones complete differentiando, ipsis etiam constantibus variabilium instar habitis, substitutione quantitatum et differentialium facta aequatio proposita transformabitur in aequationem inter novem variabiles a, b, c, ... k. Hujus autem aequationis integrale completum ex problemate (IX) derivandum est. Tumque quantitates a, b, c, ... k per variabiles x, y, z, t, u, p, q, r, s, v, exprimendo, integratio quinque aequationibus hujus formae exhibebitur:

1) $F(x, y, z, t, u, v, p, q, r, s) = \psi\,[f(x, \ldots s), \overset{1}{f}(x, \ldots s), \overset{2}{f}(x, \ldots s), \overset{3}{f}(x, \ldots s)]$,

2) $\overset{1}{F}(x, \ldots s) = \psi'\,[f(x, \ldots s)]$

3) $\overset{2}{F}\; x, \ldots s) = \psi'\,[\overset{1}{f}(x, \ldots s)]$

4) $\overset{3}{F}(x, \ldots s) - \psi'\,[\overset{2}{f}(x, \ldots s)]$

5) $\overset{4}{F}(x, \ldots s) = \psi'\,[\overset{3}{f}(x, \ldots s)]$

ex quibus, quotientes differentiales p, q, r, s, eliminatos concipiendo, prodit aequatio quaesita inter ipsas variabiles x, y, z, t, u, v.

§. 14.

Problema XI.

Aequationem differentialem inter decem variabiles per systema quinque aequationum integrare.

Solutio.

Sit aequatio proposita inter decem variabiles u, x, y, z, t, p, q, r, v, w, haec:

$$du = Pdx + Qdy + Rdz + Sdt + Tdp + Udq + Wdr + \mathfrak{X}dv + \mathfrak{Y}dw.$$

Ad quam per systema quinque aequationum integrandum requiritur, ut eadem transformetur in aequationem inter novem variabiles, quippe quam per tale systema integrabilem esse ex §. 12. constat. Haec transformatio eadem methodo perficitur, qua hactenus usi sumus. Ponatur nimirum

F

$$\begin{aligned}
dx &= X du + \chi da + \ldots + \chi^{viii} dk \\
dy &= Y du + \eta da + \ldots\ldots \\
dz &= Z du + \zeta da + \ldots\ldots \\
dt &= \mathfrak{T} du + \tau da + \ldots\ldots \\
dp &= \mathfrak{P} du + \pi da + \ldots\ldots \\
dq &= \mathfrak{Q} du + q da + \ldots\ldots \\
dr &= \mathfrak{R} du + r da + \ldots\ldots \\
dv &= \mathfrak{V} du + \mathfrak{v} da + \ldots\ldots \\
dw &= \mathfrak{W} du + \mathfrak{w} da + \ldots\ldots
\end{aligned}$$

His valoribus in aequatione proposita substitutis, prodit aequatio inter u et novem quantitates in calculum introductas a, b, . . , k. Quae nunc aequatio ab u et du liberanda est. Quem in finem formentur ratione hactenus adhibita novem aequationes conditionales, ex quibus quantitates X, Y, Z, . . $\mathfrak{W}$ tanquam functiones quantitatum x, y, . . . w, u determinare licet. Qua determinatione inventa formentur hae aequationes auxiliares

$$\begin{aligned}
dx &= X du \\
dy &= Y du \\
&\ldots\ldots \\
dw &= \mathfrak{W} du
\end{aligned}$$

ex iisque x, y, . . . w per u et novem constantes arbitrarias a, b, c, . . . k exprimantur; tumque his expressionibus complete differentiatis, constantibus etiam variatis, per substitutionem aequatio proposita transformabitur in aequationem differentialem inter novem quantitates a, b, . . . k. Quam ex problemate (IX) §. 12. integrando, tumque loco quantitatum a, b, . . . k earundem expressiones per ipsas decem variabiles x, y, . . . w, u substituendo, integratio aequationis propositae quinque aequationibus hujus formae absolvetur:

1) $F(x,y,z,t,p,q,r,v,w,u) = \psi\,[f(x,\ldots u), \overset{1}{f}(x,\ldots u), \overset{2}{f}(x,\ldots u), \overset{3}{f}(x,\ldots u)]$

2) $\overset{1}{F}(x,\ldots u) = \psi'\,[f(x,\ldots u)]$

3) $\overset{2}{F}(x,\ldots u) = \psi'\,[\overset{1}{f}(x,\ldots, u)]$

4) $\overset{3}{F}(x,\ldots u) = \psi'\,[\overset{2}{f}(x,\ldots u)]$

5) $\overset{4}{F}(x,\ldots u) = \psi'\,[\overset{3}{f}(x,\ldots u)]$.

ubi signis $F, \overset{1}{F}, \ldots \overset{4}{F}, f, \overset{1}{f}, \ldots \overset{3}{f}$, functiones notas, signo ψ functionem arbitrariam designari, nec non quomodo signum ψ' in quavis aequationum (2) — (5) accipiendum sit, ex §. 5. constat.

§. 15.

Ex casibus hactenus expositis progressus ulterior ad quotcunque variabiles satis superque manifestus est; indeque sequitur integratio completa aequationum differentiarum partialium primi ordinis inter quotcunque variabiles; nec minus evidens est, eadem methodo aequationes differentiales vulgares itidem primi ordinis inter $2m$ et $2m-1$ variabiles per systema m aequationum integrabiles esse.

Cum vero haec solutio poscat transformationem aequationis differentialis inter $2m$ variabiles in aequationem inter $2m-1$ variabiles, ostendendum restat, qua lege haec transformatio generaliter sit instituenda. Duo hic problemata discernenda videntur, alterum speciale, alterum generale. Primo quidem aequatio differentiarum partialium inter m variabiles, considerata tanquam aequatio differentialis vulgaris inter $2m-2$ variabiles ad aequationem inter $2m-3$, variabiles revocanda est. Deinde generaliter aequatio quaecunque differentialis primi ordinis inter $2m$ variabiles in aequationem inter $2m-1$ variabiles transformanda est. Priorem reductionem seorsim exponere convenit, quoniam ea calculo satis compendioso per formulas simplicissimas peragitur. Transformatio contra generalior calculos complicatiores postulat, quorum legem magis absconditam illustrare operae pretium esse videtur.

Quae modo dicta problematis duobus sequentibus absolvuntur.

§. 16.

Problema XII.

Aequationem differentiarum partialium inter $n+1$ variabiles ad aequationem differentialem vulgarem inter $2n-1$ variabiles reducere.

Solutio.

Designentur n variabiles literis $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \ldots \overset{n}{x}$, et $n+1^{ta}$, quae tanquam harum functio consideratur, littera z. Sint porro quotientes differentiales τῆ z, secundum istas variabiles accepti, $= \overset{1}{p}, \overset{2}{p}, \overset{3}{p} \ldots \overset{n}{p}$, eritque

$$dz = \overset{1}{p}d\overset{1}{x} + \overset{2}{p}d\overset{2}{x} + \overset{3}{p}d\overset{3}{x} \ldots . + \overset{n}{p}d\overset{n}{x}.$$

Jam cum detur relatio inter quotientes differentiales et ipsas variabiles, assumere licet $\overset{n}{p}$ tanquam functionem datam τῶν

$$\overset{1}{x},\ \overset{2}{x},\ \ldots\ \overset{n}{x},\ z,\ \text{et}\ \overset{1}{p},\ \overset{2}{p},\ \ldots\ \overset{n-1}{p}\ .$$

Sit igitur $\overset{n}{p} = \varphi\left(\overset{1}{x},\ \overset{2}{x},\ \overset{3}{x},\ \ldots\ \overset{n}{x},\ z,\ \overset{1}{p},\ \overset{2}{p},\ \ldots\ \overset{n-1}{p}\right)$,

tum erit ex La Grangii notandi ratione

$$d\overset{n}{p} = \varphi'\overset{1}{x}.d\overset{1}{x} + \varphi'\overset{2}{x}.d\overset{2}{x} \ldots + \varphi'\overset{n}{x}.d\overset{n}{x} + \varphi'z.dz + \varphi'\overset{1}{p}.d\overset{1}{p} \ldots + \varphi'\overset{n-1}{p}.\ d\overset{n-1}{p},$$

ubi $\varphi'\overset{1}{x}$, $\varphi'\overset{2}{x} \ldots$ itidem sunt functiones datae.

Ponamus nunc more hactenus servato

$$\begin{aligned}
d\overset{1}{x} &= \overset{1}{X}d\overset{n}{x} + \overset{1}{A}da + \overset{1}{B}db + \overset{1}{C}dc + \ldots + \overset{1}{M}dm\\
d\overset{2}{x} &= \overset{2}{X}d\overset{n}{x} + \overset{2}{A}da + \overset{2}{B}db + \overset{2}{C}dc + \ldots\\
d\overset{3}{x} &= \overset{3}{X}d\overset{n}{x} + \overset{3}{A}da + \overset{3}{B}db + \overset{3}{C}dc + \ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
d\overset{n-1}{x} &= \overset{n-1}{X}\ dx + \overset{n-1}{A}\ da + \overset{n-1}{B}\ db + \overset{n-1}{C}\ dc + \ldots\\
dz &= Zd\overset{n}{x} + \overset{1}{\zeta}da + \overset{2}{\zeta}db + \overset{3}{\zeta}dc + \ldots\\
d\overset{1}{p} &= \overset{1}{P}d\overset{n}{x} + \overset{1}{\mathfrak{A}}da + \overset{1}{\mathfrak{B}}db + \overset{1}{\mathfrak{C}}dc + \ldots\\
d\overset{2}{p} &= \overset{2}{P}d\overset{n}{x} + \overset{2}{\mathfrak{A}}da + \overset{2}{\mathfrak{B}}db + \overset{2}{\mathfrak{C}}dc + \ldots\\
d\overset{3}{p} &= \overset{3}{P}d\overset{n}{x} + \overset{3}{\mathfrak{A}}da + \overset{3}{\mathfrak{B}}db + \overset{3}{\mathfrak{C}}dc + \ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
d\overset{n-1}{p} &= \overset{n-1}{P}\ d\overset{n}{x} + \overset{n-1}{\mathfrak{A}}\ da + \overset{n-1}{\mathfrak{B}}\ db + \overset{n-1}{\mathfrak{C}}\ dc + \ldots
\end{aligned}$$

tum aequatio proposita in hanc abit:

$$0 = \left.\begin{array}{r}\overset{1}{p}\overset{1}{X}\\ +\overset{2}{p}\overset{2}{X}\\ +\overset{3}{p}\overset{3}{X}\\ \ldots\\ +\overset{n-1}{p}\ \overset{n-1}{X}\\ +\overset{n}{p}\\ -Z\end{array}\right| d\overset{n}{x}
\left.\begin{array}{r}+\overset{1}{p}\overset{1}{A}\\ +\overset{2}{p}\overset{2}{A}\\ \\ \ldots\\ +\overset{n-1}{p}\ \overset{n-1}{A}\\ \\ -\overset{1}{\zeta}\end{array}\right| da
\left.\begin{array}{r}+\overset{1}{p}\overset{1}{B}\\ +\overset{2}{p}\overset{2}{B}\\ \\ \ldots\\ +\overset{n-1}{p}\ \overset{n-1}{B}\\ \\ -\overset{2}{\zeta}\end{array}\right| db
\left.\begin{array}{r}+\overset{1}{p}\overset{1}{C}\\ +\overset{2}{p}\overset{2}{C}\\ \\ \ldots\\ +\overset{n-1}{p}\ \overset{n-1}{C}\\ \\ -\overset{3}{\zeta}\end{array}\right| dc + \ldots$$

Quo nunc haec aequatio a $d\overset{n}{x}$ et ab $\overset{n}{x}$ liberetur, poni debet primo

$$Z = \overset{1}{p}\overset{1}{X} + \overset{2}{p}\overset{2}{X} + \overset{3}{p}\overset{3}{X} \ldots + \overset{n-1}{p}\overset{n-1}{X} + \overset{n}{p}.$$

Deinde quotiens $\dfrac{d\overset{n}{x}(\overset{1}{p}\overset{1}{A}+\overset{2}{p}\overset{2}{A}\ldots+\overset{n-1}{p}\overset{n-1}{A}-\overset{1}{\zeta})}{\overset{1}{p}\overset{1}{A}+\overset{2}{p}\overset{2}{A}\ldots+\overset{n-1}{p}\overset{n-1}{A}-\overset{1}{\zeta}}$

invariatus manere debet, litteram A cum B, C, D, ... nec non $\overset{1}{\zeta}$ cum $\overset{2}{\zeta}, \overset{3}{\zeta}, \overset{4}{\zeta}$ etc. permutando. Est autem, brevitatis caussa differentialia secundum $\overset{n}{x}$ simpliciter littera d designando, $d(\overset{1}{p}\overset{1}{A}+\overset{2}{p}\overset{2}{A}\ldots+\overset{n-1}{p}\overset{n-1}{A}-\overset{1}{\zeta})$

$$=\overset{1}{p}d\overset{1}{A}+\overset{2}{p}d\overset{2}{A}\ldots+\overset{n-1}{p}d\overset{n-1}{A}-d\overset{1}{\zeta}+\overset{1}{A}d\overset{1}{p}+\overset{2}{A}d\overset{2}{p}\ldots+\overset{n-1}{A}d\overset{n-1}{p}$$

$$=\overset{1}{p}d^2\overset{1}{X}+\overset{2}{p}d^2\overset{2}{X}\ldots+\overset{n-1}{p}d^2\overset{n-1}{X}-d^2Z+\overset{1}{A}d\overset{1}{p}+\overset{2}{A}d\overset{2}{p}\ldots+\overset{n-1}{A}d\overset{n-1}{p}$$

$$=\left\{\begin{array}{l} d^2(\overset{1}{X}\overset{1}{p}+\overset{2}{X}\overset{2}{p}\ldots+\overset{n-1}{X}\overset{n-1}{p}-Z) \\ -\overset{1}{X}d^2\overset{1}{p}-\overset{2}{X}d^2\overset{2}{p}\ldots-\overset{n-1}{X}d^2\overset{n-1}{p}+\overset{1}{A}d\overset{1}{p}+\overset{2}{A}d\overset{2}{p}\ldots+\overset{n-1}{A}d\overset{n-1}{p}\end{array}\right.$$

$$=-d^2\overset{n}{p}-\overset{1}{X}d^2\overset{1}{p}-\overset{2}{X}d^2\overset{2}{p}\ldots\overset{n-1}{X}d^2\overset{n-1}{p}+\overset{1}{A}d\overset{1}{p}+\overset{2}{A}d\overset{2}{p}\ldots+\overset{n-1}{A}d\overset{n-1}{p}.$$

Est autem $d^2\overset{1}{p}=\overset{1}{\mathfrak{A}}$, $d^2\overset{2}{p}=\overset{2}{\mathfrak{A}}$, ..., $d^2\overset{n-1}{p}=\overset{n-1}{\mathfrak{A}}$;

$$d^2\overset{n}{p}=\left\{\begin{array}{l}\varphi'\overset{1}{x}.d^2\overset{1}{x}+\varphi'\overset{2}{x}.d^2\overset{2}{x}\ldots+\varphi'\overset{n-1}{x}.d^2\overset{n-1}{x}+\varphi'z.d^2z \\ +\varphi'\overset{1}{p}.d^2\overset{1}{p}+\varphi'\overset{2}{p}.d^2\overset{2}{p}\ldots+\varphi'\overset{n-1}{p}.d^2\overset{n-1}{p}\end{array}\right.$$

$$=\left\{\begin{array}{l}\varphi'\overset{1}{x}.\overset{1}{A}+\varphi'\overset{2}{x}.\overset{2}{A}\ldots+\varphi'\overset{n-1}{x}.\overset{n-1}{A}+\varphi'z.\overset{1}{\zeta} \\ \varphi'\overset{1}{p}.\overset{1}{\mathfrak{A}}+\varphi'\overset{2}{p}.\overset{2}{\mathfrak{A}}\ldots+\varphi'\overset{n-1}{p}.\overset{n-1}{\mathfrak{A}};\end{array}\right.$$

porro $d\overset{1}{p}=\overset{1}{P}$, $d\overset{2}{p}=\overset{2}{P}$, $d\overset{3}{p}=\overset{3}{P}$, ... $d\overset{n-1}{p}=\overset{n-1}{P}$.

Hinc fit $d(\overset{1}{p}\overset{1}{A}+\overset{2}{p}\overset{2}{A}+\ldots+\overset{n-1}{p}\overset{n-1}{A}-\overset{1}{\zeta})=$

$$\left\{\begin{array}{l}
\begin{array}{r|r|r|l} -\varphi'\overset{1}{x} & \overset{1}{A}-\varphi'\overset{2}{x} & \overset{2}{A}\ldots-\varphi'\overset{n-1}{x} & \overset{n-1}{A}-\varphi'z.\overset{1}{\zeta} \\ +\overset{1}{P} & +\overset{2}{P} & +\overset{n-1}{P} & \end{array} \\
\begin{array}{r|r|r|l} -\varphi'\overset{1}{p} & \overset{1}{\mathfrak{A}}-\varphi'\overset{2}{p} & \overset{2}{\mathfrak{A}}\ldots-\varphi'\overset{n-1}{p} & \overset{n-1}{\mathfrak{A}} \\ -\overset{1}{X} & -\overset{2}{X} & -\overset{n-1}{X} & \end{array}
\end{array}\right.$$

Conditio igitur quoad praedictum quotientem adimplebitur, si ponatur:

$$1)\quad \overset{1}{X} = -\varphi'\overset{1}{p}$$

$$2)\quad \overset{2}{X} = -\varphi'\overset{2}{p}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$n-1)\quad \overset{n-1}{X} = -\varphi'\overset{n-1}{p}$$

$$n)\quad \overset{1}{P} - \varphi'\overset{1}{x} = \varphi'z\;\overset{1}{p}$$

$$\text{sive}\quad \overset{1}{P} = \varphi'\overset{1}{x} + \overset{1}{p}\,.\,\varphi'z$$

$$n+1)\quad \overset{2}{P} = \varphi'\overset{2}{x} + \overset{2}{p}\varphi'z$$

$$n+2)\quad \overset{3}{P} = \varphi'\overset{3}{x} + \overset{3}{p}\varphi'z$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$2n-1)\quad \overset{n-1}{P} = \varphi'\overset{n-1}{x} + \overset{n-1}{p}\,.\,\varphi'z.$$

Hinc fit ex aequatione priori

$$2n-1)\quad Z = \overset{n}{p} - \overset{1}{p}\varphi'\overset{1}{p} - \overset{2}{p}\varphi'\overset{2}{p} \ldots - \overset{n-1}{p}\,\varphi'\overset{n-1}{p}.$$

Quare aequationes differentiales auxiliares hae sunt:

$$1)\quad d\overset{1}{x} = -\varphi'\overset{1}{p}\,.\,d\overset{n}{x}$$

$$2)\quad d\overset{2}{x} = -\varphi\overset{2}{p}\;d\overset{n}{x}$$

$$3)\quad d\overset{3}{x} = -\varphi\overset{3}{p}\,.\,d\overset{n}{x}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$n-1)\quad d\overset{n-1}{x} = -\varphi'\overset{n-1}{p}\,.\,d\overset{n}{x}$$

$$n)\quad dz = \overset{n}{p} - \overset{1}{p}\varphi'\overset{1}{p} - \overset{2}{p}\varphi'\overset{2}{p} \ldots - \overset{n-1}{p}\,\varphi'\overset{n-1}{p})\,.\,dx$$

$$n+1)\quad d\overset{1}{p} = (\varphi\overset{1}{x} + \overset{1}{p}\varphi'z)\,d\overset{n}{x}$$

$$n+2)\quad d\overset{2}{p} = (\varphi\overset{2}{x} + \overset{2}{p}\varphi'z)\,d\overset{n}{x}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$2n-1)\quad d\overset{n-1}{p} = (\varphi\overset{n-1}{x} + \overset{n-1}{p}\,\varphi'z)\,d\overset{n}{x}$$

Ex quibus integrando $\overset{1}{x}, \overset{2}{x}, \ldots \overset{n-1}{x}, z, \overset{1}{p}, \overset{2}{p}, \ldots \overset{n-1}{p}$ per $\overset{n}{x}$ et $2n-1$ constantes arbitrarias a, b, c, e, ... exprimi debent (§. 2.). Quas deinceps expressiones complete differentiando, (constantibus etiam variabilium instar habitis), et in aequatione proposita substituendo, haec in aequationem differentialem vulgarem inter praedictas $2n-1$ quantitates a, b, c, ..., e, abit.

§. 17.

Problema XIII.

Aequationem differentialem vulgarem primi ordinis inter $2m$ variabiles in aequationem similem inter $2m-1$ variabiles transformare.

Solutio.

Ex solutionibus supra pro 4, 6, 8, 10 variabilibus traditis constat, $2m-1$ variabiles aequationis differentialis propositae tanquam functiones $2m^{tae}$ et $2m-1$ novarum quantitatum, illarum loco introducendarum, exprimendas esse. Quas quidem novas quantitates pro constantibus habendo, prodeunt $2m-1$ aequationes differentiales auxiliares, quarum integratio completa ipsas functiones desideratas suppeditat. At vero ad formandas hasce aequationes auxiliares requiruntur $2m-1$ quantitates, quarum valores per totidem aequationes conditionales determinantur. Haec determinatio, si consueta eliminandi methodo tractetur, calculos nimium complicatos et operosos postulat; ipsaque praecepta generalia, quae **Bezout** et **Cramer** de eliminatione tradiderunt, in casu substrato parum commodi afferre videntur. Accuratius vero considerando praedictas aequationes conditionales et formulas ex earum solutione actu evolutas, ad duas leges satis simplices easque generales perveni, quas hic breviter exponere sufficiat [h]).

Lex prima formationis aequationum differentialium auxiliarium

Inchoemus a casu primo quatuor variabilium, seu ab aequatione $dz = Pdx + Qdy + Rdp$, pro qua supra §. 4. evolvimus has aequationes auxiliares:

$$\frac{dy}{dz} = \frac{PR^{v} - RP^{iv} + R' - P'''}{PQ''' - P'''Q + QR' - Q'R + RP'' - PR''},$$

$$\frac{dp}{dz} = \frac{QP^{iv} - PQ^{iv} + P'' - Q'}{PQ''' - P'''Q + QR' - Q'R + RP'' - PR''},$$

$$\frac{dx}{dz} = \frac{RQ^{iv} - QR^{v} + Q''' - R''}{PQ''' - P'''Q + QR' - Q'R + RP'' - PR''}.$$

h) Harum legum, calculis haud parum molestis confirmatarum, demonstrationem nimis quidem prolixam hic omitto, quanquam breviorem eam reddi posse haud dubito.

Quo lex generalis evidentior fiat, aequatio differentialis proposita sub hac forma exhibeatur: $o = A da + B db + C dc + E de$, ubi litterae a, b, c, e cum iis, per quas supra constantes ex integratione aequationum auxiliarium ingressas notavimus, haud permiscendae, indeque hae constantes aliis litteris e.g. $\alpha, \beta, \gamma, \ldots$ vel $\dot{a}, \dot{b}, \dot{c}, \ldots$ designandae sunt. Tum erunt $x, y, p, z = a, b, c, e$;

$$P = -\frac{A}{E},\ Q = -\frac{B}{E},\ R = -\frac{C}{E};$$

hinc $PQ''' - P'''Q = \frac{PdQ - QdP}{dp} = \frac{P^2 d\frac{Q}{P}}{dp} = \frac{A^2}{E^2} \cdot \frac{d\frac{B}{A}}{dc} = \frac{AdB - BdA}{E^2 dc}$;

simili modo est $QR' - Q'R = \frac{BdC - CdB}{E^2 da}$;

$RP'' - PR'' = \frac{CdA - AdC}{E^2 db}$; $PR^{IV} - RP^{IV} = \frac{AdC - CdA}{E^2 de}$;

$$R' - P''' = \frac{dR}{dx} - \frac{dP}{dp} = \frac{-EdC + CdE}{E^2 da} + \frac{EdA - AdE}{E^2 dc}.$$

$$\text{Hinc fit}\quad \frac{db}{de} = \frac{\frac{AdC - CdA}{de} + \frac{CdE - EdC}{da} + \frac{EdA - AdE}{dc}}{\frac{AdB - BdA}{dc} + \frac{BdC - CdB}{da} + \frac{CdA - AdC}{db}}.$$

Simili modo exprimere licet $\frac{dc}{de}$ et $\frac{da}{de}$. Quod si nunc omisso termino $B db$ fingatur aequatio: $o = A da + C dc + E de$, notum est, conditionem integrabilitatis hujus aequationis, tanquam aequationis inter tres variabiles, duas independentes supponendo, hac aequatione exprimi: (Euler Calc. Integr. Vol. III. p. 6.)

$$o = \frac{AdC - CdA}{de} + \frac{CdE - EdC}{da} + \frac{EdA - AdE}{dc}.$$

Hanc formulam integrabilitatis (quam hoc loco tantum compendii caussa in auxilium vocamus, tanquam expressionem analyticam notam et formatu facilem), designemens hoc charactere: (ACE), ubi ad ordinem litterarum A, C, E, respi-

respiciendum est, ita quidem ut sit

$$(AEC) = \frac{AdE - EdA}{dc} + \frac{EdC - CdE}{da} + \frac{CdA - AdC}{de} = -(ACE),$$

et $$(ECA) = \frac{EdC - CdE}{da} + \frac{CdA - AdC}{de} + \frac{AdE - EdA}{dc} = -(ACE).$$

Simili modo denominator fractionis, qua $\frac{db}{de}$ exprimitur, signo (ABC) notandus est; indeque prodit $\frac{db}{de} = \frac{(ACE)}{(ABC)}$, sive $o = \frac{db}{(ACE)} + \frac{de}{(ACB)}$.

In hac aequatione permutando invicem e et c, E et C (quoniam termini aequationis differentialis propositae ad lubitum transponi possunt), erit

$$o = \frac{db}{(AEC)} + \frac{dc}{(AEB)}, \text{ sive } o = \frac{db}{(ACE)} + \frac{dc}{(ABE)};$$

deinde in priori aequatione permutando e et a, E et A, fit

$$o = \frac{db}{(ECA)} + \frac{da}{(ECB)}, \text{ sive } o = \frac{db}{(ACE)} + \frac{da}{(BCE)}.$$

Inde aequationes tres auxiliares hanc formam induunt:

1) $$o = \frac{db}{(ACE)} + \frac{dc}{(ABE)}$$

2) $$o = \frac{db}{(ACE)} + \frac{de}{(ACB)}$$

3) $$o = \frac{db}{(ACE)} + \frac{da}{(BCE)},$$

ubi observare licet, quod ex denominatore τῦ db, oriantur denominatores τῶν dc; de; et da; ponendo B pro C; E; A. Eaedem aequationes sub hac forma etiam exhiberi possunt:

1) $$o = \frac{da}{(BCE)} + \frac{dc}{(BAE)}$$

2) $$o = \frac{da}{(BCE)} + \frac{de}{(BCA)}$$

3) $$o = \frac{da}{(BCE)} + \frac{db}{(ACE)}.$$

ubi nunc ex denominatore τῷ da prodeunt denominatores τῶν dc; de; db, ponendo A pro C; E; B.

Transeamus nunc ad aequationem inter sex variabiles hanc:

$$o = A da + B db + C dc + E de + F df + G dg,$$

ex supra §. 8. demonstratis, sponte sequitur aequatio auxiliaris inter db et dg haec:

$$\frac{db}{dg} = \frac{(ACF).(AFG) - (ACF).(AEG) + (ACG).(AEF)}{(ABC).(AEF) - (ABE).(ACF) + (ABF).(ACE)},$$

$$\text{sive} \left\{ \begin{array}{l} \dfrac{db}{(ACE)(AFG) - (ACF)(AEG) + (ACG)(AEF)} \\ + \dfrac{dg}{(ACE)(AFB) - (ACF).(AEB) + (ACB)(AEF)} \end{array} \right. .$$

Permutando invicem a et b, A et B, fit

$$o = \left\{ \begin{array}{l} \dfrac{da}{(BCE)(BFG) - (BCF)(BEG) + (BCG)(BEF)} \\ + \dfrac{dg}{(BCE)(BFA) - (BCF)(BEA) + (BCA)(BEF)} \end{array} \right.$$

Permutando hîc invicem c et g, C et G, fit

$$o = \left\{ \begin{array}{l} \dfrac{da}{(BGE)(BFC) - (BGF)(BEC) + (BGC)(BEF)} \\ + \dfrac{dc}{(BGE)(BFA) - (BGF)(BEA) + (BGA)(BEF)} \end{array} \right.$$

$$\text{sive } o = \left\{ \begin{array}{l} \dfrac{da}{(BCE)(BFG) - (BCF)(BEG) + (BCG)(BEF)} \\ + \dfrac{dc}{(BAE)(BFG) - (BAF)(BEG) + (BAG)(BEF)} \end{array} \right.$$

Simili modo aequatio inter da, de; et da, df reperitur. Quare hae quatuor aequationes auxiliares prodeunt:

$$1)\ o = \left\{ \begin{array}{l} \dfrac{da}{(BCE)(BFG) - (BCF)(BEG) + (BCG)(BEF)} \\ + \dfrac{dc}{(BAE)(BFG) - (BAF)(BEG) + (BAG)(BEF)} \end{array} \right.$$

$$2)\quad 0 = \left\{ \begin{array}{l} \dfrac{da}{(BCE)(BFG) - (BCF)(BEG) + (BCG)(BEF)} \\ + \dfrac{de}{(BCA)(BFG) - (BCF)(BAG) + (BCG)(BAF)} \end{array} \right.$$

$$3)\quad 0 = \left\{ \begin{array}{l} \dfrac{da}{(BCE)(BFG) - (BCF)(BEG) + (BCG)(BEF)} \\ + \dfrac{df}{(BCE)(BAG) - (BCA)(BEG) + (BCG)(BEA)} \end{array} \right.$$

$$4)\quad 0 = \left\{ \begin{array}{l} \dfrac{da}{(BCE)(BFG) - (BCF)(BEG) + (BCG)(BEF)} \\ + \dfrac{dg}{(BCE)(BFA) - (BCF)(BEA) + (BCA)(BEF)} \end{array} \right.$$

ubi ex denominatore τῦ da prodeunt denominatores τῶν dc; de; df; dg, ponendo A pro C; E; F; G. Haec lex exceptionem patitur pro aequatione differentiali inter da et db; quae ipsa autem aequatio sponte derivatur ex qualibet quatuor praecedentium, e. g. ex prima, si c et b, C et B inter se invicem permutentur, unde fit

$$5)\quad 0 = \left\{ \begin{array}{l} \dfrac{da}{(CBE)(CFG) - (CBF)(CEG) + (CBG)(CEF)} \\ + \dfrac{db}{(CAE)(CFG) - (CAF)(CEG) + (CAG)(CEF)} \end{array} \right.$$

Simili quidem ratione etiam aequationes (2), (3), (4) ex (1) derivare licet, verum ratio differentiae in eo cernitur, quod permutando c cum e, f, g; C cum E, F, G, denominator τῦ da signum tantum mutet, permutando autem c et b, ipse denominator immutetur. Quare modus supra dictus deducendi aequationes (2), (3), (4) simplicior videtur.

Progrediamur ad aequationem inter octo variabiles hanc:

$$0 = A\,da + B\,db + C\,dc + E\,de + F\,df + G\,dg + H\,dh + I\,di,$$

tum erunt aequationes auxiliares hae:

$$1)\quad 0 = \frac{da}{\mathfrak{A}} + \frac{dc}{\mathfrak{C}},$$

$$2)\quad 0 = \frac{da}{\mathfrak{A}} + \frac{de}{\mathfrak{E}},$$

$$3)\quad o = \frac{da}{\mathfrak{A}} + \frac{df}{\mathfrak{F}},$$

$$4)\quad o = \frac{da}{\mathfrak{A}} + \frac{dg}{\mathfrak{G}},$$

$$5)\quad o = \frac{da}{\mathfrak{A}} + \frac{dh}{\mathfrak{H}},$$

$$6)\quad o = \frac{da}{\mathfrak{A}} + \frac{di}{\mathfrak{I}},$$

$$7)\quad o = \frac{da}{\mathfrak{A}} + \frac{db}{\mathfrak{B}},$$

Lex supra expressa hic etiam observatur, ut nimirum denominatores τῶν dc, de, df, dg, dh, di, sive $\mathfrak{C}, \mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{I}$, ex denominatore $\mathfrak{A}$ τοῦ da oriantur, ponendo A pro C, E, F, G, H, I. Septima autem aequatio ex prima oritur, permutando in hac inter se invicem c et b, C et B. Restat igitur tantum, ut determinetur denominator $\mathfrak{A}$, ejusque fomandi lex eruatur. Reperitur autem, calculis rite subductis, factores communes aequationum auxiliarium omittendo, et terminos se mutuo destruentes delendo,

$$\mathfrak{A} = \left\{\begin{array}{l} (BCE)(BFG)(BHI) - (BCE)(BFH)(BGI) + (BCE)(BFI)(BGH) \\ - (BCF)(BEG)(BHI) + (BCF)(BEH)(BGI) - (BCF)(BEI)(BGH) \\ + (BCG)(BEF)(BHI) - (BCG)(BEH)(BFI) + (BCG)(BEI)(BFH) \\ - (BCH)(BEF)(BGI) + (BCH)(BEG)(BFI) - (BCH)(BEI)(BFG) \\ + (BCI)(BEF)(BGH) - (BCI)(BEG)(BFH) + (BCI)(BEH)(BFG). \end{array}\right.$$

Separando litteram B, termini hujus expressionis complectuntur permutationes litterarum reliquarum C, E, F, G, H, I, (exclusa prima A), quae sub hac restrictione fieri possunt, ut litterae in quavis complexione (ex. gr. C, G, E, H, F, I in termino octavo τοῦ $\mathfrak{A}$) prima, tertia, quinta (e. gr. C, E, F), in genere imparem locum obtinentes inter se rite sint ordinatae, et litterarum quaevis pari loco constituta (G, H, I) sit ordine alphabetico posterior littera in loco impari proxime praecedente (C, E, F). His formis rite inter se ordinatis, i. e. secundum ordinem lexicographicum (e. g. C, G, E, H, F, I ante C, G, E, I, F, H), terminorum signa alternant. Haec lex restrictiva permutationum etiam sic enuntiari potest, ut singulas complexiones dispertiendo in dyades, sive classes binorum elementorum, ipsae dyades tam quoad sua elementa, quam inter se invicem rite debeant esse ordinatae.

Eadem nunc lege generaliter pro quotcunque litteris, vel pro aequationibus inter quotcunque variabiles, denominator τῦ da formatur; cumque lex, qua reliqui denominatores ex hoc deducuntur, etiam constanter valeat, ratione exposita aequationes auxiliares universaliter formare licet. Processus autem combinatorius, quo permutationes praedictae exhibentur, satis commodus hic est: Sint litterae, quarum permutationes sub restrictione supra commemorata quaeruntur a, b, c, e, ... k, l, m, n; supponamus inventas esse permutationes litterarum c, e, ... m, n, exclusis duabus a, b: tum 1) singulis his permutationibus vel complexionibus praeponatur binio ab; 2) ex hac prima serie complexiones totidem aliae formentur, permutando b et c; 3) ex his porro aliae, permutando c et d, sicque progrediendo ex quavis serie complexionum nova formetur, litteram aliquam cum proxime sequente permutando, donec postremo m et n invicem permutentur. Qua ratione obtinentur omnes permutationes litterarum a, b, c, ... m, n, quas restrictio praedicta admittit. Manifestum est, inchoando a litteris m, n, ab his ad k, l progrediendo, et sic porro, hoc modo tandem permutationes quaesitas sub forma involutoria (ex Hindenburgii appellatione) reperiri.

Notatu dignum videtur, quod multitudo permutationum praedictarum inter 2n elementa per productum numerorum imparium $1 . 3 . 5 \ldots (2n-1)$ exprimatur, cum ex formula vulgo nota numerus omnium permutationum possibilium sit $= 1 . 2 . 3 . 4 \ldots 2n.$ [i]) Sic pro aequatione differentiali inter decem variabiles (§. 14.) hac:

$$o = A da + B db + C dc + E de + F df + G dg + H dh + I di + K dk + L dl,$$

denominator $\mathfrak{A}$ terminis constat $105 = 1 . 3 . 5 . 7$, quorum evolutionem brevitatis caussa hic omitto.

i) Obiter addo propositionem combinatoriam generaliorem hanc. Si $\alpha . n$ elementa sub hac restrictione inter se permutentur, ut singulas complexiones in classes α elementorum dispertiendo, hae classes tam quoad elementa sua, quam inter se invicem rite sint ordinatae, tum erit numerus permutationum $= \frac{(n+1)(n+2) \ldots \alpha n}{(1 . 2 . 3 \ldots \alpha)^n}$. Pro $\alpha = 2$ observandum est, esse $\frac{(n+1)(n+2) \ldots (2n+1)}{1 . 3 . 5 \ldots 2n+1} = 2^n$, unde pro hoc casu particulari expressio simplicior per productum numerorum imparium prodit.

§. 18.

Continuatio.

Altera lex formationis aequationum differentialium auxiliarium.

Formulae hactenus exhibitae denominatorum 𝔄, 𝔅, ℭ, ... constant terminis, qui ipsi sunt producta plurium factorum non simplicium, sed ex sex partibus compositorum. Ita v. c. factor (BCE) est =

$$\frac{BdC - CdB}{de} + \frac{CdE - EdC}{db} + \frac{EdB - BdE}{dc} =$$

$$BC^{iv} - CB^{iv} + CE'' - EC'' + EB''' - BE'''.$$

Quotientes nimirum differentiales quantitatum A, B, C, ... more supra observato indicibus distinguimus, e. g. $\frac{dC}{da}$ est $= C'$, $\frac{dC}{db} = C''$, $\frac{dC}{dc} = C'''$, $\frac{dC}{de} = C^{iv}$, etc. ita quidem, ut litteris tam majoribus quam minoribus (C, c) indices tribuendo numericos secundum ordinem alphabeticum, index numericus litterae majori superscriptus designet ejusdem quotientem differentialem, dum ea littera minor tenquam variabilis consideretur, cui idem index competit. Ita ex. gr. est $C^{iv} = \frac{dC}{de}$, quoniam index numericus litterae e est $= 4$.

Quod si nunc ista producta ex factoribus compositis actuali multiplicatione evolvantur, expressiones praedictae denominatorum alias formas induunt, quarum termini nunc ex productis factorum simplicium constant. Pro aequatione differentiali inter quatuor variabiles

$$o = Ada + Bdb + Cdc + Ede,$$

aequationes auxiliares ex supra demonstratis hae sunt:

$$1)\ o = \begin{cases} \dfrac{da}{BC^{iv} - CB^{iv} + CE'' - EC'' + EB''' - BE'''} \\ + \dfrac{db}{AC^{iv} - CA^{iv} + CE' - EC' + EA''' - AE'''} \end{cases}$$

sive terminos aliter ordinando,

$$o = \left\{ \begin{array}{l} \dfrac{da}{B(C^{iv} - E''') - C(B^{iv} - E'') + E(B''' - C'')} \\ + \dfrac{db}{A(C^{iv} - E''') - C(A^{iv} - E') + E(A''' - C')} \end{array} \right.$$

$$2)\ o = \left\{ \begin{array}{l} \dfrac{da}{B(C^{iv} - E''') - C(B^{iv} - E'') + E(B''' - C'')} \\ + \dfrac{dc}{B(A^{iv} - E') - A(B^{iv} - E'') + E(B' - A'')} \end{array} \right.$$

$$3)\ o = \left\{ \begin{array}{l} \dfrac{da}{B(C^{iv} - E''') - C(B^{iv} - E'') + E(B''' - C'')} \\ + \dfrac{de}{B(C' - A''') - C(B' - A'') + A(B''' - C'')} \end{array} \right.$$

In tribus his aequationibus denominatores τῶν db, dc, de, prodeunt ex denominatore τοῦ da, ponendo A pro B, C, E; simulque indicem τοῦ A sive I pro indicibus τῶν B, C, E; sive pro II, III, IV. Haec lex, ex primo denominatore τοῦ da ceteros derivandi, uniformiter pro quotcunque litteris a, b, c, e, f, g, h etc. valet, et quidem pro omnibus aequationibus auxiliaribus, absque ulla exceptione, quae in superiori regula formationis (§. 17.) pro aequatione inter da et db observanda erat. Quare explicandum duntaxat est, quomodo formandus sit denominator τοῦ da.

Pro aequatione inter sex variabiles

$$o = Ada + Bdb + Cdc + Ede + Fdf + Gdg,$$

sint aequationes auxiliares:

$$1)\ o = \frac{da}{\mathfrak{A}} + \frac{db}{\mathfrak{B}}.$$

$$2)\ o = \frac{da}{\mathfrak{A}} + \frac{dc}{\mathfrak{C}}$$

$$3)\ o = \frac{da}{\mathfrak{A}} + \frac{de}{\mathfrak{E}}$$

$$4)\ o = \frac{da}{\mathfrak{A}} + \frac{df}{\mathfrak{F}}$$

$$5)\ o = \frac{da}{\mathfrak{A}} + \frac{dg}{\mathfrak{G}}.$$

tum erit denominator

$$\mathfrak{A} = \left\{ \begin{array}{l} B\left(\begin{array}{l} C^{v}E^{vi} - C^{iv}F^{vi} + C^{iv}G^{v} - E^{v}C^{vi} + E'''F^{vi} - E'''G^{v} \\ + F^{iv}C^{vi} - F'''E^{vi} + F'''G^{iv} - G^{iv}C^{v} + G'''E^{v} - G'''F^{iv} \end{array}\right) \\ - C\left(\begin{array}{l} B^{v}E^{vi} - B^{iv}F^{vi} + B^{iv}G^{v} - E^{v}B^{vi} + E''F^{vi} - E''G^{v} + F^{iv}B^{vi} \\ - F''E^{vi} + F''G^{iv} - G^{iv}B^{v} + G''E^{v} - G''F^{iv} \end{array}\right) \\ + E\left(\begin{array}{l} B^{v}C^{vi} - B'''F^{vi} + B'''G^{v} - C^{v}B^{vi} + C''F^{vi} - C''G^{v} \\ + F'''B^{vi} - F''C^{vi} + F''G''' - G'''B^{v} + G''C^{v} - G''F''' \end{array}\right) \\ - F\left(\begin{array}{l} B^{iv}C^{vi} - B'''E^{vi} + B'''G^{iv} - C^{iv}B^{vi} + C''E^{vi} - C''G^{iv} \\ + E'''B^{vi} - E''C^{vi} + E''G''' - G'''B^{iv} + G''C^{iv} - G''E''' \end{array}\right) \\ + G\left(\begin{array}{l} B^{iv}C^{v} - B'''E^{v} + B'''F^{iv} - C^{iv}B^{v} + C''E^{v} - C''F^{iv} \\ + E'''B^{v} - E''C^{v} + E''F''' - F'''B^{iv} + F''C^{iv} - F''E''' \end{array}\right). \end{array} \right.$$

Hae expressiones, si ad solas litteras B, C, E, F, G, respicimus, complectuntur omnes earum terniones cum permutationibus, sive variationum in doctrina combinatoria sic dictarum tertiam classem et quidem rite ordinatam. Quomodo signa se habeant, manifestum est: ea nimirum alternant duplici respectu, primo quoad factores B, C, E, F, G, deinde quoad singulos terminos, qui in hos factores ducti sunt. Quod ad indices numericos litterarum attinet, in quavis complexione vel in quovis producto ii junguntur indices ordine naturali, qui litteris in hoc producto deficientibus (exclusa semper littera A) competunt, prima littera cujusvis producti indice caret. Ex. gr. in productis BEG, BGE, EBG, EGB, GBE, GEB adduntur secundo et tertio factori indices litterarum C. F in his productis non occurrentium, sive numeri III, V. Simili modo pro aequatione differentiali inter octo variabiles:

$$o = A\,da + B\,db + C\,dc + E\,de + F\,df + G\,dg + H\,dh + I\,di,$$

denominator $\mathfrak{A}$ complectitur omnes quaterniones septem litterarum B, C, E, F, G, H, I cum permutationibus; pro aequatione inter 10 variabiles:

$$o = A\,da + B\,db + \ldots + K\,dk + L\,dl,$$

denominator comprehendit omnes conquinationes cum permutationibus novem litterarum B, C, ... K, L. Generatim pro aequatione differentiali inter 2n variabiles, denominator τῦ da sive $\mathfrak{A}$ constat omnibus variationibus 2n — 1 litterarum B, C, E, ... (exclusa prima A) classis n^tae^. Signa terminorum, (dum classis rite sit ordinata, sive singulae complexiones procedant ordine lexico-

gra-

graphico), et indices numerici litteris jungendi sequuntur legem praedictam: nec minus ex ante dictis, constat, quomodo reliqui denominatores ex primo 𝔄 deriventur. Transitum a prima regula formationis (§. 17.) ad hanc alteram, hujusque vim sequens exemplum monstrabit. Pro aequatione differentiali inter octo variabiles, denominator τῶ da ex regula posteriore constat terminis $7 . 6 . 5 . 4 = 840$, qui numerus multitudinem quaternionum cum permutationibus ex septem elementis exprimit. Secundum regulam priorem denominator τῶ da constat terminis tantum 15, at si producta trium factorum, quorum quivis sex partibus constat, evolvantur, quodvis productum praebet $6 . 6 . 6 = 216$ partes, quare numerus terminorum denominatoris post hanc evolutionem assurgit ad $15 . 216 = 3240$, quos nunc vi regulae secundae ad 840 contrahi, reliquis 2400 se mutuo destruentibus, certum est. Idque ipsum evolutione actu instituta confirmatur.

§ 19

Quanquam methodus hactenus exposita, aequationem differentialem primi ordinis inter $2n$ variabiles ad aequationem similem inter $2n-1$ variabiles reducendi, ad quemvis variabilium numerum etiam imparem eadem ratione extendi posse videatur, accuratior tamen consideratio contrarium ostendit, atque docet, aequationem inter $2n+1$ variabiles non generaliter ad $2n$ variabiles reduci posse, sed tum demum hanc reductionem locum habere, si certa relatio inter coefficientes aequationis differentialis obtineat. Quod phaenomenon notatu dignum duobus exemplis illustrasse sufficiat.

Sit aequatio proposita inter tres variabiles haec: $dz = Pdx + Qdy$, ubi P, Q sunt functiones datae τῶν x, y, z. Quod si nunc methodo hactenus adhibita hanc aequationem ad aequationem inter duas variabiles revocare tentemus, ponendum est

$$dx = Xdz + \chi da + \chi' db$$
$$dy = Ydz + \eta da + \eta' db,$$

tum aequatio proposita in hanc abit:

$$0 = \left.\begin{matrix} PX \\ +QY \\ -1 \end{matrix}\right| dz + \left.\begin{matrix} P\chi \\ +Q\eta \end{matrix}\right| da + \left.\begin{matrix} P\chi' \\ +Q\eta' \end{matrix}\right| db.$$

Quae ut a dz et z liberetur, poni debet

1) $PX + QY = 1$;

2) $d^z\left(\frac{P\chi' + Q\eta'}{P\chi + Q\eta}\right) = 0$ sive $\frac{d^z(P\chi + Q\eta)}{P\chi + Q\eta} = \frac{d^z(P\chi' + Q\eta')}{P\chi' + Q\eta'}$.

Est autem $d^z(P\chi + Q\eta) = \left\{\begin{array}{l} P d^z\chi + Q d^z\eta \\ + \chi d^z P + \eta d^z Q \end{array}\right.$

$$= \left\{\begin{array}{l} = P d^a X + Q d^a Y \\ + \chi d^z P + \eta d^z Q \end{array}\right. = \left\{\begin{array}{l} d^a(PX + QY) - X d^a P - Y d^a Q \\ \quad + \chi d^z P + \eta d^z Q \end{array}\right.$$

$$= - X d^a P - Y d^a Q + \chi d^z P + \eta d^z Q.$$

Cum sint P et Q functiones τῶν x, y, et z, sit

$$dP = P' dx + P'' dy + P''' dz$$
$$dQ = Q' dx + Q'' dy + Q''' dz,$$

et erit $d^a P = P' d^z x + P'' d^a y = P'\chi + P''\eta$,

$d^z P = P' d^z x + P'' d^z y + P''' = P' X + P'' Y + P'''$.

Simili modo, permutando P cum Q, exprimuntur $d^a Q$, $d^z Q$.

Hinc fit $$\frac{d^z(P\chi + Q\eta)}{P\chi + Q\eta} = \frac{\left.\begin{array}{l} P'X \\ + P''Y \\ + P''' \\ - XP' \\ - YQ' \end{array}\right\}\chi \quad \left.\begin{array}{l} + Q'X \\ + Q''Y \\ + Q''' \\ - XP'' \\ - YQ'' \end{array}\right\}\eta}{P\chi + Q\eta}$$

$$= \frac{[(P'' - Q')Y + P''']\chi + [(Q' - P'')X + Q''']\eta}{P\chi + Q\eta}.$$

Eadem ratione permutando χ, η, cum χ', η' exprimitur quotiens $\frac{d^z(P\chi' + Q\eta')}{P\chi' + Q\eta'}$,

qui primo erit aequalis, si ponatur $\frac{(Q' - P'')X + Q'''}{Q} = \frac{(P'' - Q')Y + P'''}{P}$, sive

$$(Q' - P'')PX - (P'' - Q')QY + PQ''' - P'''Q = 0.$$

Quae aequatio ad determinandas duas incognitas X, Y jungenda est priori (1) $PX + QY = 1$. At sumendo ex hac $PX = 1 - QY$, et substituendo in altera aequatione, ex hac ipsa exit Y, fit enim

$$Q - P' - Q(Q' - P'')Y - (P'' - Q')QY + PQ''' - P'''Q = o,$$
$$\text{i. e.} \quad Q' - P' + PQ''' - P'''Q = o.$$

Haec est aequatio conditionalis, sive relatio, quae inter coefficientes P, Q aequationis propositae intercedere debet, quo ea ad aequationem inter duas variabiles reduci queat.

Haec aequatio conditionalis praebet, signa usitata adhibendo,

$$o = \frac{dQ}{dx} - \frac{dP}{dy} + \frac{PdQ}{dz} - \frac{QdP}{dz},$$

quod ipsum apprime conspirat cum noto criterio integrabilitatis aequationis $dz = Pdx + Qdy$, consideratae tanquam aequationis inter tres variabiles z, x, y, quarum una esse debet functio reliquarum duarum nullo inter se nexu analytico junctorum (§. 17.). Quod autem aequatio differentialis inter tres variabiles hoc sensu integrabilis seu illi criterio satisfaciens, ad aequationem inter duas variabiles revocari queat, aliunde demonstrare licet.

Simili modo tractetur aequatio differentialis inter quinque variabiles haec:

$$dv = Pdx + Qdy + Rdz + Sdu,$$

ubi P, Q, R, S, sunt functiones datae τῶν x, y, z, u, v. Ad transformandam hanc aequationem in aequationem inter quatuor variabiles a, b, c, e, ponamus:

$$\begin{aligned} dx &= Xdu + \chi da + \chi' db + \chi'' dc + \chi''' de, \\ dy &= Ydu + \eta da + \eta' db + \eta'' dc + \eta''' de, \\ dz &= Zdu + \zeta da + \zeta' db + \zeta'' dc + \zeta''' de, \\ dv &= Vdu + \mathfrak{v} da + \mathfrak{v}' db + \mathfrak{v}'' dc + \mathfrak{v}''' de; \end{aligned}$$

tunc abit aequatio proposita in hanc:

$$o = \begin{array}{r|r|r|r|r|l} V & du \quad + \mathfrak{v} & da \quad + \mathfrak{v}' & db \quad + \mathfrak{v}'' & dc \quad + \mathfrak{v}''' & de \\ -PX & -P\chi & -P\chi' & -P\chi'' & -P\chi''' & \\ -QY & -Q\eta & -Q\eta' & -Q\eta'' & -Q\eta''' & \\ -RZ & -R\zeta & -R\zeta' & -R\zeta'' & -R\zeta''' & \\ -S & & & & & \end{array}$$

Quae aequatio ut a du et u liberetur, poni debet

1) $V = PX + QY + RZ + S;$

Deinde quotiens $\frac{d^{u}(\mathfrak{v}-P\chi-Q\eta-R\zeta)}{\mathfrak{v}-P\chi-Q\eta-R\zeta}$ idem manere debet, ponendo pro

$$\mathfrak{v},\ \chi,\ \eta,\ \zeta;\quad \mathfrak{v}',\ \chi',\ \eta',\ \zeta';\quad \mathfrak{v}'',\ \chi'',\ \eta'',\ \zeta'';\quad \mathfrak{v}''',\ \chi''',\ \eta''',\ \zeta'''.$$

Est autem $d^{u}(\mathfrak{v}-P\chi-Q\eta-R\zeta)=$

$$d^{u}\mathfrak{v}-Pd^{u}\chi-Qd^{u}\eta-Rd^{u}\zeta-\chi d^{u}P-\eta d^{u}Q-\zeta d^{u}R$$

$$=d^{a}V-Pd^{a}X-Qd^{a}Y-Rd^{a}Z-\chi d^{u}P-\eta d^{u}Q-\zeta d^{u}R$$

$$=\begin{cases} d^{a}(V-PX-QY-RZ)+Xd^{a}P+Yd^{a}Q+Zd^{a}R \\ -\chi d^{u}P-\eta d^{u}Q-\zeta d^{u}R \end{cases}$$

$$=d^{a}S+Xd^{a}P+Yd^{a}Q+Zd^{a}R-\chi d^{u}P-\eta d^{u}Q-\zeta d^{u}R.$$

Quum P, Q, R, S, sint datae functiones τῶν x, y, z, u, v, ponamus

$$dP=P'dx+P''dy+P'''dz+P^{iv}du+P^{v}dv,$$

et simili modo exprimamus dQ, dR, dS.

Tum erit $d^{a}P=P'\chi+P''\eta+P'''\zeta+P^{v}\mathfrak{v}$,

$$d^{u}P=P'X+P''Y+P'''Z+P^{iv}+P^{v}V,$$

et similes expressiones obtinentur pro reliquis differentialibus secundum a et u.

Quas substituendo fit $d^{u}(\mathfrak{v}-P\chi-Q\eta-R\zeta)=$

S'	χ	$+S''$	η	$+S'''$	ζ	$+S^{v}$	$\mathfrak{v}$
$+YQ'$		$+XP''$		$+XP'''$		$+XP^{v}$	
$+ZR'$		$+ZR''$		$+YQ'''$		$+YR^{v}$	
$-P''Y$		$-Q'X$		$-R'X$		$+ZR^{v}$	
$-P'''Z$		$-Q'''Z$		$-R''Y$			
$-P^{v}V$		$-Q^{v}V$		$-R^{v}V$			
$-P^{iv}$		$-Q^{iv}$		$-R^{v}$			

Quare praedicta conditio quoad quotientem $\frac{d^{u}(\mathfrak{v}-P\chi-Q\eta-R\zeta)}{\mathfrak{v}-P\chi-Q\eta-R\zeta}$ adimplebitur, si hasce tres aequationes assumamus:

$$2)\ \begin{cases} -P(S^{v}+P^{v}X+Q^{v}Y+R^{v}Z) \\ =S'+YQ'+ZR'-P''Y-P'''Z-P^{iv}-P^{v}V \end{cases}$$

$$3)\ \begin{cases} -Q(S^{v}+P^{v}X+Q^{v}Y+R^{v}Z) \\ =S''+XP''+ZR''-Q'X-Q'''Z-Q^{iv}-Q^{v}V \end{cases}$$

$$4)\ \begin{cases} -R(S^{v}+P^{v}X+Q^{v}Y+R^{v}Z) \\ =S'''+XP'''+YQ'''-R'X-R''Y-R^{iv}-R^{v}V. \end{cases}$$

Quibus aequationibus junctis priori (1), quatuor quantitates incognitae

X, Y, Z, V determinari videntur. At vero calculo actu evolvendo atque ad finem perducendo, eliminatio, incognita ex calculo exeunte, deducit ad aequationem conditionalem hanc:

$$o = \begin{cases} (Q^{v}S - QS^{v} + Q^{iv} - S'')\ (PR^{v} - P^{v}R + R' - P'') \\ -\ (P^{v}S - PS^{v} + P^{iv} - S')\ (QR^{v} - Q^{v}R + R'' - Q''') \\ -\ (R^{v}S - RS^{v} + R^{iv} - S''')\ (PQ^{v} - P^{v}Q + Q' - P''). \end{cases}$$

Quare aequatio proposita inter quinque variabiles tum demum ad aequationem inter quatuor variabiles reduci poterit, cum haec relatio inter ejus coefficientes locum habeat. Simili modo pro septem, novem etc. variabilibus ratiocinari licet. Legem generalem aequationis conditionalis pro $2n+1$ variabilibus, quo aequatio differentialis ad $2n$ variabiles revocari queat, (quae quidem lex ad similitudinem formularum supra §. 17, 18. expositarum exprimi potest), nec non reliqua consectaria ex hac observatione singulari petenda nunc praetermitto.

§. 20.

Problema XIV.

Integrationem completam aequationum differentiarum partialium ad formam simpliciorem revocare.

Solutio.

Sit $dz = \overset{1}{p}d\overset{1}{x} + \overset{2}{p}d\overset{2}{x} + \overset{3}{p}d\overset{3}{x} \ldots + \overset{n}{p}d\overset{n}{x}$ (§. 16.), et supponatur data relatio inter variabiles $z, \overset{1}{x}, \overset{2}{x}, \ldots \overset{n}{x}$, et quotientes differentiales $\overset{1}{p}, \overset{2}{p}, \ldots \overset{n}{p}$, cujus ope $\overset{n}{p}$ per reliquas harum quantitatum exprimere liceat. Tum ex praecedenti expositione constat, integrationem completam hujus aequationis differentiarum partialium aequationibus hujus formae exhiberi:

$$1)\ F(z, \overset{1}{x}, \overset{2}{x}, \ldots \overset{n}{x}, \overset{1}{p}, \overset{2}{p}, \ldots \overset{n-1}{p}) = \psi \begin{Bmatrix} f(z, \overset{1}{x} \ldots \overset{n}{x}, \overset{1}{p}, \ldots \overset{n-1}{p}), \\ \overset{1}{f}(z, \overset{1}{x} \ldots \overset{n-1}{p}), \overset{2}{f}(z \ldots \overset{n-1}{p}), \\ \ldots \overset{n-2}{f}(z, \overset{1}{x}, \ldots \overset{n-1}{p}) \end{Bmatrix}$$

$$2)\ \overset{1}{F}(z, \overset{1}{x} \ldots \overset{n-1}{p}) = \psi' [f(z, \overset{1}{x} \ldots \overset{n-1}{p})]$$

$$3)\ \overset{2}{F}(z, \overset{1}{x}, \ldots \overset{n-1}{p}) = \psi' [\overset{1}{f}(z, \overset{1}{x}, \ldots \overset{n-1}{p})]$$

. . . .

$$n)\ \overset{n-1}{F}(z, \overset{1}{x}, \ldots \overset{n-1}{p}) = \psi' [\overset{n-2}{f}(z, \overset{1}{x}, \ldots \overset{n-1}{p})]$$

Ponamus jam $f(z, \overset{1}{x}, \overset{1}{x}, \ldots \overset{n-1}{p}) = \overset{1}{k}$,

$$\overset{1}{f}(z, \overset{1}{x}, \ldots \overset{n-1}{p}) = \overset{1}{k},$$
$$\overset{2}{f}(z, \overset{1}{x}, \ldots \overset{n-1}{p}) = \overset{3}{k},$$
$$\ldots\ldots\ldots$$
$$\overset{n-1}{f}(z, \overset{1}{x}, \ldots \overset{n-1}{p}) = \overset{n-1}{k}$$

tum concipere licet, ope harum $n-1$ aequationum quantitates $p, \overset{1}{p}, \ldots \overset{n-1}{p}$ expressas esse per $z, \overset{1}{x}, \ldots \overset{n}{x}$, et per $\overset{1}{k}, \overset{2}{k}, \ldots \overset{n-1}{k}$. Quo facto functiones signis $F, \overset{1}{F}, \overset{2}{F}, \ldots \overset{n-1}{F}$ denotatae etiam abeunt in functiones cognitas earundem quantitatum $z, \overset{1}{x}, \ldots \overset{n}{x}, \overset{1}{k}, \overset{2}{k}, \ldots \overset{n-1}{k}$ quas functiones signis $\mathfrak{F}, \overset{1}{\mathfrak{F}}, \overset{2}{\mathfrak{F}}, \overset{3}{\mathfrak{F}}, \ldots \overset{n-1}{\mathfrak{F}}$ exprimamus. Quare aequationes integrales hanc formam nanciscentur:

1) $\mathfrak{F}(z, \overset{1}{x}, \overset{2}{x}, \ldots \overset{n}{x}, \overset{1}{k}, \overset{2}{k}, \ldots \overset{n-1}{k}) = \psi(\overset{1}{k}, \overset{2}{k}, \ldots \overset{n-1}{k})$

2) $\overset{1}{\mathfrak{F}}(z, \overset{1}{x}, \ldots \overset{n-1}{k}) = \psi' \overset{1}{k}$

3) $\overset{2}{\mathfrak{F}}(z, \overset{1}{x}, \ldots \overset{n-1}{k}) = \psi' \overset{2}{k}$

4) $\overset{3}{\mathfrak{F}}(z, \overset{1}{x}, \ldots \overset{n-1}{k}) = \psi' \overset{3}{k}$

$\ldots\ldots\ldots$

n) $\overset{n-1}{\mathfrak{F}}(z, \overset{1}{x}, \ldots \overset{n-1}{k}) = \psi' \overset{n-1}{k}$.

Nunc vero inter functiones per $\mathfrak{F}, \overset{1}{\mathfrak{F}}, \overset{2}{\mathfrak{F}}, \ldots \overset{n-1}{\mathfrak{F}}$ designatas, simplex et memorabilis intercedit relatio, cujus ope ex prima functione reliquas facile determinare licet: ad quam quidem relationem perductus sum accuratiori consideratione nexus inter aequationes integrales et aequationem differentialem propositam, seu modi, quo illae huic satisfaciant.

Differentiando nimirum aequationem (1) obtinetur:

$$dz.\mathfrak{F}'z + d\overset{1}{x}.\mathfrak{F}'\overset{1}{x} + d\overset{2}{x}.\mathfrak{F}'\overset{2}{x}\ldots + d\overset{1}{k}.\mathfrak{F}'\overset{1}{k} + d\overset{2}{k}.\mathfrak{F}'\overset{2}{k}\ldots + d\overset{n-1}{k}.\mathfrak{F}'\overset{n-1}{k}$$
$$= d\overset{1}{k}.\psi'\overset{1}{k} + d\overset{2}{k}.\psi'\overset{2}{k}\ldots + d\overset{n-1}{k}.\psi'\overset{n-1}{k}.$$

Hinc fit, substituendo pro $\psi'\overset{1}{k}, \psi'\overset{2}{k}, \ldots \psi'\overset{n-1}{k}$, valores ex aequationibus (2), (3), ... (n) prodeuntes, quos brevitatis caussa solis litteris functionalibus $\overset{1}{\mathfrak{F}}, \overset{2}{\mathfrak{F}}, \overset{3}{\mathfrak{F}}, \dots \overset{n-1}{\mathfrak{F}}$ designemus,

$$o = \begin{cases} dz \cdot \mathfrak{F}'z + d\overset{1}{x} \cdot \mathfrak{F}'\overset{1}{x} + d\overset{2}{x} \cdot \mathfrak{F}'\overset{2}{x} \dots + d\overset{n}{x} \cdot \mathfrak{F}'\overset{n}{x} \\ + d\overset{1}{k}(\mathfrak{F}'\overset{1}{k} - \overset{1}{\mathfrak{F}}) + d\overset{2}{k}(\mathfrak{F}'\overset{2}{k} - \overset{2}{\mathfrak{F}}) \dots + d\overset{n-1}{k}(\mathfrak{F}'\overset{n-1}{k} - \overset{n-1}{\mathfrak{F}}). \end{cases}$$

Haec aequatio, jam a functione arbitraria liberata, identica esse debet cum aequatione differentiali proposita $dz = \overset{1}{p} d\overset{1}{x} + \overset{2}{p} d\overset{2}{x} \dots \overset{n}{p} d\overset{n}{x}$.

Qui consensus manifesto necessarius est, si consideremus totam expositionem supra traditam integrationis aequationum differentialium inter quotcunque variabiles, quibus etiam nostra aequatio differentiarum partialium adnumeranda est. Quodsi enim numerus variabilium in aequatione differentiali proposita impar sit, tum consensus sive identitas praedicta immediate ex praecedentibus sequitur, quia differentiatio aequationis integralis primae, substituendo valores ex reliquis aequationibus, producit ipsam aequationem differentialem propositam. Idem vero etiam de numero pari variabilium valere, inde efficitur, quod aequatio differentialis inter $2n$ variabiles in aliam ipsi omnino aequipollentem inter $2n-1$ variabiles transformetur. Quibus praemissis hae deducuntur aequationes identicae:

$$\mathfrak{F}'\overset{1}{k} = \overset{1}{\mathfrak{F}}$$
$$\mathfrak{F}'\overset{2}{k} = \overset{2}{\mathfrak{F}}$$
$$\mathfrak{F}'\overset{3}{k} = \overset{3}{\mathfrak{F}}$$
$$\dots\dots$$
$$\mathfrak{F}'\overset{n-1}{k} = \overset{n-1}{\mathfrak{F}}$$

Quare functiones signis $\overset{1}{\mathfrak{F}}, \overset{2}{\mathfrak{F}}, \dots \overset{n-1}{\mathfrak{F}}$ denotatae ex prima functione littera $\mathfrak{F}$ insignita deriventur, hanc differentiando secundum $\overset{1}{k}, \overset{2}{k}, \dots \overset{n-1}{k}$. Inde aequationes integrales sub hac forma simplici exhibere licet:

1) $\mathfrak{F}(z, \overset{1}{x}, \overset{2}{x}, \dots \overset{n}{x}, \overset{1}{k}, \overset{2}{k}, \dots \overset{n-1}{k}) = \psi(\overset{1}{k}, \overset{2}{k}, \dots \overset{n-1}{k})$

2) $\mathfrak{F}'(\overset{1}{k}) = \psi'(\overset{1}{k})$

3) $\mathfrak{F}'(\overset{2}{k}) = \psi'(\overset{2}{k})$

4) $\mathfrak{F}'(\overset{3}{k}) = \psi'(\overset{3}{k})$

. . . .

n) $\mathfrak{F}'(\overset{n-1}{k}) = \psi'(\overset{n-1}{k})$

Ex his aequationibus solutio completa aequationis differentiarum partialium statim atque immediate sub forma generali deduci potest. Namque functionem arbitrariam per ψ expressam indeterminatam relinquendo, ex variabilibus $n + 1$, quas aequatio proposita involvit, $(z, \overset{1}{x}, \ldots \overset{n}{x})$, n variabiles per $n + 1^{\text{tam}}$ et per quantitates $n - 1$ indeterminatas $\overset{1}{k}, \overset{2}{k}, \ldots \overset{n-1}{k}$, ope istarum n aequationum integralium exprimi possunt. Tales quidem expressiones generales, ipsam functionem arbitrariam involventes, forma prior aequationum integralium suppeditare nequit. Namque ex his aequationibus quantitates $\overset{1}{p}, \overset{2}{p}, \ldots \overset{n}{p}$ tum demum eliminare, indeque aequationem inter ipsas variabiles $z, \overset{1}{x}, \ldots \overset{n}{x}$, eruere licet, cum loco functionis arbitrariae signo ψ denotatae certa atque determinata functio supponatur, quod quovis casu solutionem tantum particularem evolutam praebet.

Ceterum ex forma simpliciori hic exhibita etiam quotientes differentiales $\overset{1}{p}, \overset{2}{p}, \ldots \overset{n}{p}$ per easdem n quantitates indeterminatas exprimere licet. Cum enim aequatio $o = dz . \mathfrak{F}'z + d\overset{1}{x} . \mathfrak{F}'\overset{1}{x} \ldots + d\overset{n}{x} . \mathfrak{F}'\overset{n}{x}$ sit identica cum aequatione $dz = \overset{1}{p} d\overset{1}{x} + \overset{2}{p} d\overset{2}{x} \ldots + \overset{n}{p} d\overset{n}{x}$, esse debet

$$\overset{1}{p} = \frac{-\mathfrak{F}'\overset{1}{x}}{\mathfrak{F}'z},\ \overset{2}{p} = \frac{-\mathfrak{F}'\overset{2}{x}}{\mathfrak{F}'z},\ \ldots \ldots \overset{n}{p} = \frac{-\mathfrak{F}'\overset{n}{x}}{\mathfrak{F}'z}.$$

Applicationem methodi generalis in praecedentibus expositae ad exempla ipsam magis illustrantia, varias inde natas observationes, expositionem methodorum magis specialium, quibus certos casus facilius expedire licet, (ita etiam methodi paullo diversae, aequationes differentiarum partialium inter quatuor variabiles integrandi, in quam incideram, priusquam methodo generali ad quotcunque variabiles patente potitus eram), haec et alia, ne nimis longa fiat haec commentatio, nunc praetermitto.
